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A Hybrid Single Tone Frequency Estimator

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Abstract – We propose a hybrid single real tone frequency estimator based on the Generalized Reformed Pisarenko Harmonic Decomposer method and on filtering the data sequence in order to increase the signal-to-noise ratio. We show by experiments that the proposed estimator behaves well at low signal-to-noise ratios. For *N* samples long data sequences, the complexity of the algorithm is *0*(*N*).

Keywords: frequency estimation, Generalized Reformed Pisarenko's method

I. INTRODUCTION

The problem of frequency estimation of a noisy sinusoid arises in many fields where signal processing applies, such as communications, measurements, speech analysis, radar, sonar etc. When both in-phase and quadrature components of the signal are available, dealing with complex exponentials is preferred [1, 2, 3].

The real case is also important [4, 5], and new solutions have been proposed in the last years, [6]. The Cramer-Rao Lower Bound (CRLB) has been derived and the maximum likelihood (ML) solution to this estimation problem has been tackled in [4]. Exact form of the ML estimate for arbitrary data record lengths has been published in [5]. The computational complexity rules out the ML from practical estimators. As ML estimates are not computationally efficient, other estimation algorithms are preferred, such as the Pisarenko Harmonic Decomposer (PHD) [7]. Research is targeted to finding estimators with reduced necessary data record length, decreased computational complexity, lower variance, and good performance at low signal-to-noise ratios (*SNR*'s).

Starting from the PHD, a new, significantly better estimation algorithm, named Reformed PHD (RPHD) has been proposed quite recently (see [6] and the references cited therein). Our own work showed that, if the multiple of the frequency is estimated instead of the frequency itself, the estimator variance decreases. We illustrated this point in [8] for the RPHD, and we call the result Generalized RPHD (GRPHD). We have also introduced an iterative frequency estimation procedure where an iteration consists of estimating the frequency by means of the RPHD from a given data sequence, filtering the data sequence with a frequency selective, noise rejection filter with a passband centered at the previously estimated frequency, and using the filtered data sequence for a new estimation [9]. We showed that the estimator variance approached the CRLB after a few RPHD-filtering iterations, at low *SNR*'s (3 dB).

In this paper we show that similar good results can be obtained by a combination of the GRPHD, that provides a first estimate, followed by a filtering operation with a bandpass filter with frequency response centered at the first estimate, and an application of the RPHD to the data sequence at the filter output for finding the final estimate.

II. THEORY

We consider the following signal model

$$x(n) = \alpha\cos(\omega_0 n + \varphi) + q(n)\,, \quad n = 1, 2, ..., N \qquad (1)$$

where $\alpha>0$, $\omega_0 \in (0, \pi)$ and $\varphi \in [0, 2\pi)$ are deterministic but unknown constants, which represent the amplitude, (angular) frequency and phase of the sinusoid respectively. The noise $q(n)$ is assumed to be a zero-mean, gaussian, white process with unknown variance σ^2. We have $SNR = \dfrac{\alpha^2}{2\sigma^2}$.

The GRPHD method [8] looks for an estimator $\hat{\omega}_{0k}$ of an integer multiple $k\omega_0$ of the frequency ω_0 through:

$$\rho_k = \cos(\hat{\omega}_{0k}) \qquad (2)$$

where

$$\rho_k = \frac{B_{kN} + \sqrt{B_{kN}^2 + 8A_{kN}^2}}{4A_{kN}} \qquad (3)$$

$$A_{kN} = \sum_{n=2k+1}^{N} \big(x(n) + x(n-2k)\big)x(n-k) \qquad (4)$$

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$$B_{kN} = \sum_{n=2k+1}^{N}\left((x(n)+x(n-2k))^2 - 2x^2(n-k)\right) \quad (5)$$

Equation (2) has k solutions when the argument of the cosine in the RHS is replaced by kw and it is solved for w, namely

$$w_j = \frac{1}{k}\left((-1)^{j-1}\cos^{-1}(\rho_k) + \left[\frac{j}{2}\right]2\pi\right), \quad j = 1, 2, ..., k \quad (6)$$

where we have denoted by $[x]$ the largest integer smaller than x. The estimate $\hat{\omega}_{0g} = \frac{\hat{\omega}_{0k}}{k}$ of ω_0 is obtained from (6) by making

$$j = 1 + \left[\frac{k\omega_0}{\pi}\right]. \quad (7)$$

For small k (k<<N) and for a good estimation (small $|\hat{\omega}_0 - \omega_0|$), we have [8]

$$\mathrm{var}(\hat{\omega}_{0g}) \approx \frac{\mathrm{var}(\rho_k)}{k^2 \sin^2(k\omega_0)}. \quad (8)$$

The GRPHD is a two-step method. In the first step we compute the frequency estimate $\hat{\omega}_{01}$ using (2..6), for k=1, which is basically the RPHD. For the second step we select k in the range 1..K, K<<N, in order to minimize the variance in (8) by maximizing the denominator in its RHS. Then, we compute the frequency estimate $\hat{\omega}_{0g}$ using again (2..6), with the selected k, and substituting $\hat{\omega}_{01}$ for ω_0 in (7).

The obtained frequency estimate plays the role of center frequency of a bandpass filter used to filter the initial data sequence $x(n)$ in order to increase the SNR.

We have considered the second-order noise rejection filter with the following transfer function:

$$H(z) = \frac{z^2}{z^2 - 2\rho\cos(\omega_r)z + \rho^2} \quad (9)$$

where ρ is a parameter close to unity in order to provide selectivity (but smaller, for stability), and $\omega_r = \hat{\omega}_{0g}$. The performance of this noise-rejection filter is evaluated by the enhancement η of the SNR [9]

$$\eta = \frac{SNR_1}{SNR} \approx \frac{1}{1-\rho}, \quad (10)$$

where SNR_1 is defined at the filter output.

We finally apply to the signal from the filter output the RPHD in order to calculate the frequency estimator $\hat{\omega}_0$. Interestingly enough, although the theory for the RPHD assumes a white noise and the noise after filtering gets colored, the estimator remains valid, as shown experimentally in the next section (see also [9]).

III. EXPERIMENTAL RESULTS AND COMPLEXITY EVALUATION

Computer experiments have been performed in order to evaluate the proposed estimator by comparing it with the estimators obtained through the RPHD and the GRPHD.

After the estimation in the first step of the GRPHD, which gives $\hat{\omega}_{01}$, the value of k must be chosen. The simplest solution is to use a look-up table that gives the value of k when a quantized version of $\hat{\omega}_{01}$ is entered. For the construction of the look-up table we have chosen K, we have calculated the denominator in (8) for $1 \le k \le K$ and a sufficiently refined frequency grid, and determined, for each frequency in the grid, the value of k for which the denominator in (8) was maximum. The value $K = 6$ proved experimentally to be the best choice.

The data sequence was generated using (1) with $\alpha = \sqrt{2}$, $\varphi = 0$, and $SNR = 3\ dB$. The experimentally obtained estimator mean square error versus frequency is represented in Fig. 1. The results are averages of 500 runs, with a data record length $N = 100$ and a filter parameter $\rho = 0.95$. Fig. 1 shows how the estimated frequency variance decreases. The largest variance corresponds to the RPHD, which is necessary as the first step in the GRPHD. The variance of the GRPHD estimator is smaller, as expected. After filtering and a second application of the RPHD, the final result is obtained (filtered GRPHD - FGRPHD). Note that the CRLB, also represented in Fig. 1, is approached at less than 5 dB.

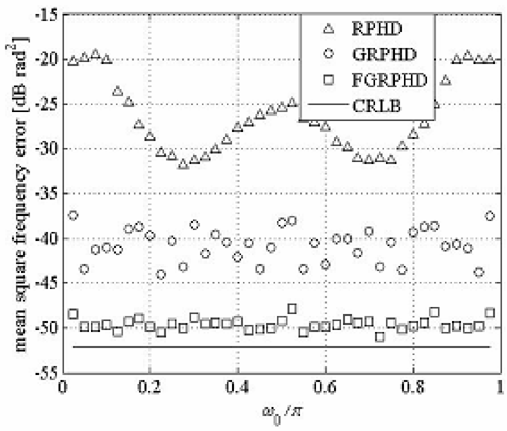

Fig.1 Mean square frequency errors versus frequency for SNR=3 dB, N=100, ρ=0.95, K=6.

The application of the proposed procedure tends to attenuate the strong dependence on frequency of the variance that is exhibited by estimators of frequencies of real sinusoids (see e.g. [6]).

In order to evaluate the calculation complexity, we will distinguish between additions, multiplications and other operations such as square roots, $\cos^{-1}$, powers, rounding to nearest, smaller integer, multiplications by powers of two, choosing k, which are implemented by look-up tables, shifts or other means. The numbers of each operation are summarized in Table 1.

Table 1. Calculation complexity

	RPHD	GRPHD	Filtering	Total
Real Additions	$4N$-8	$4N$-$8k$+2	$2N$-3	$14N$-$8k$-17
Real Multipl.	$3N$-4	$3N$-$6k$+3	$2N$-2	$11N$-$6k$-7
Other	5	9	0	19
Times	2	1	1	

IV. CONCLUSIONS

We have proposed a hybrid frequency estimation method, with a complexity of $0(N)$ additions and multiplications, consisting of a combination of existing estimators, two of which we have introduced in other works. We have proved by experiments that the proposed method behaves well at signal-to-noise ratios as low as 3 dB, approaching by less than 5 dB the *CRLB*, which is around -50 dB in this case.

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