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A Three-Point Interpolated DFT Method For Frequency Estimation

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Abstract – In this paper a three-point interpolated discrete Fourier transform (IpDFT) method for estimating with high accuracy the frequency of a multifrequency signal component is presented. The performance of the proposed method has been analyzed by means of computer simulations for a multifrequency signal without noise and with quantization noise as well.

Index terms: frequency estimation, interpolated DFT method, Hann window.

I. INTRODUCTION

In a real case the coherent frequency relationship between all the frequencies contained in a multifrequency signal and the sampling frequency are not meet, leading to the well-known leakage phenomena. The way used to reduce this problem is called 'windowing' and a frequency-domain method often used for estimating the frequency of a multifrequency signal component under noncoherent sampling is the interpolated DFT (IpDFT) method [1]-[4]. This method provides very accurate frequency estimates. The IpDFT method with Hann window leads to very accurate estimates because permits to estimate the frequency of a multifrequency signal component by analytical formula [2]. In this paper a three-point IpDFT method for estimating with high accuracy the frequency of a multifrequency signal component is proposed. The performance of the proposed method is analyzed by means of computer simulations for a multifrequency signal without noise and with quantization noise, respectively.

II. FREQUENCY ESTIMATION

Let us consider a multifrequency signal sampled at f_s frequency:

$$x(m) = A_0 + \sum_{k=1}^{K} A_k \sin\left(2\pi \frac{f_k}{f_s} m + \varphi_k\right) \qquad m = 0, 1, \ldots, M-1 \qquad (1)$$

where K is the number of frequency components, A_k, f_k and φ_k are respectively the amplitude, frequency and phase of the k^{th} component of the multifrequency signal, A_0 is the offset of the multifrequency signal and M is the number of samples acquired. The Discrete-Time Fourier Transform (DTFT) of $x_w(m) = x(m)\cdot w(m)$ is given by

$$X_w(\lambda) = \sum_{m=0}^{M-1} x(m) w(m) e^{-j2\pi\lambda \frac{m}{M}}, \qquad \lambda \in [0, M) \qquad (2)$$

where λ represents the normalized frequency expressed in bin.

After some calculus $X_w(\lambda)$ becomes

$$X_w(\lambda) = \sum_{m=0}^{M-1} \frac{A_k}{2j} \left[W(\lambda - \lambda_k) e^{j\varphi_k} - W(\lambda + \lambda_k) e^{-j\varphi_k} \right], \qquad \lambda \in [0, M) \qquad (3)$$

where W(λ) is the DTFT of the window and $\lambda_k = f_k/f_0$, in which $f_0 = f_s/M$.

If W(λ) exhibits sidelobes with negligible level and if the minimum distance between spectral lines is more larger than MLBW (MainLobeBandWidth) expressed in bin, then for $\lambda \cong \lambda_k$ we have

$$X_w(\lambda) = \frac{A_k}{2j} W(\lambda - \lambda_k) e^{j\varphi_k}, \quad k = 1, 2, \ldots, K. \qquad (4)$$

Since the frequencies f_k and f_s does not fulfill the coherent frequency relationship we have

$$\frac{f_k}{f_s} = \frac{\lambda_k}{M} = \frac{l_k + \delta_k}{M}, \qquad k = 1, 2, \ldots, K \qquad (5)$$

where $\lambda_k = l_k + \delta_k$, in which l_k is the number of the recorded k^{th} component cycles (l_k is an integer) and δ_k is the fractional part of the recorded of the k^{th} component cycles, $-0.5 \le \delta_k < 0.5$.

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Thus, from (4) and (5) it follows

$$|X_w(l_k - 1)| = 0.5A_k|W(1+\delta_k)| \tag{6a}$$

$$|X_w(l_k)| = 0.5A_k|W(-\delta_k)| \tag{6b}$$

$$|X_w(l_k + 1)| = 0.5A_k|W(1-\delta_k)| \tag{6c}$$

with the observation that the maximum in the discrete spectrum computed by DFT corresponding to the k^{th} component is located at $l_k f_0$.

The windows employed in the proposed method is the Hann window, defined by

$$w(m) = 0.5 - 0.5\cos\left(\frac{2\pi m}{M}\right),\ m = 0, 1, \ldots, M-1 \tag{7}$$

For $M >> 1$, the DTFT of the window $w(m)$ can be approximated by

$$W(\lambda) = \frac{M\sin(\pi\lambda)}{2\pi\lambda(1-\lambda^2)} e^{-j\pi\lambda} e^{j\frac{\pi}{M}\lambda} \tag{8}$$

Denoting by α_k the rapport

$$\alpha_k = \frac{|X_w(l_k+1)|^2 - |X_w(l_k-1)|^2}{|X_w(l_k-1)|^2 + |X_w(l_k)|^2 + |X_w(l_k+1)|^2} \tag{9}$$

From the expression (6) the rapport α_k is given by

$$\alpha_k = \frac{|W(1-\delta_k)|^2 - |W(1+\delta_k)|^2}{|W(1+\delta_k)|^2 + |W(\delta_k)|^2 + |W(1-\delta_k)|^2} \tag{10}$$

Based on the expression (8) after some calculus the rapport α_k becomes

$$\alpha_k = \frac{4\delta_k}{\delta_k^2 + 4} \tag{11}$$

The above equation has two solutions

$$\delta_{k1,2} = \frac{2 \pm 2\sqrt{1-\alpha_k^2}}{\alpha_k} \tag{12}$$

Since the first solution is higher than 2 then the only possibility is

$$\delta_k = \frac{2 - 2\sqrt{1-\alpha_k^2}}{\alpha_k} \tag{13}$$

Based on the relationship (5) the frequency f_k is given by

$$f_k = (l_k + \delta_k) f_0 \tag{14}$$

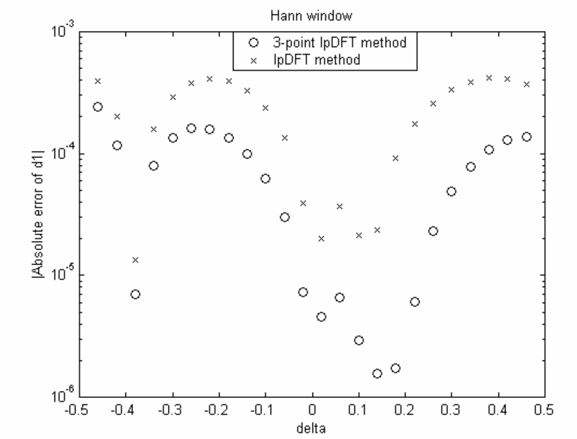
III. COMPUTER SIMULATION

The effectiveness of the proposed method is analyzed by means of computer simulation. First the analysis is made in the case of a multifrequency signal without noise. The signal used in this case in simulation is

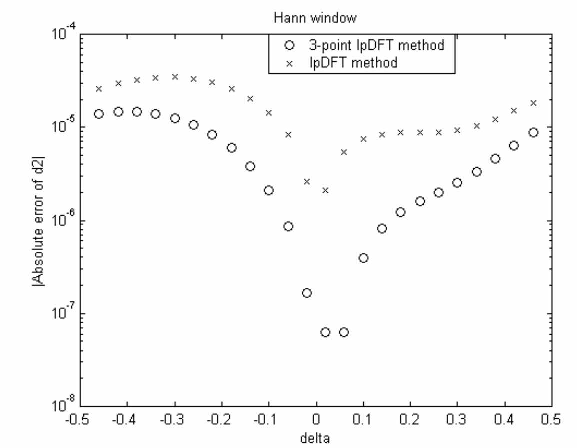
$$\begin{aligned} x(m) = A_0 &+ A_1 \sin\left(2\pi\frac{l_1+\delta_1}{M}m + \varphi_1\right) \\ &+ A_2 \sin\left(2\pi\frac{l_2+\delta_2}{M}m + \varphi_2\right) \\ &+ A_3 \sin\left(2\pi\frac{l_3+\delta_3}{M}m + \varphi_3\right) \\ &+ A_4 \sin\left(2\pi\frac{l_4+\delta_4}{M}m + \varphi_4\right) \end{aligned} \tag{15}$$

in which $A_0 = 0.1$, $A_1 = 2$, $A_2 = 0.5$, $A_3 = 0.07$, $A_4 = 0.1$, $\varphi_1 = 0.4$ rad, $\varphi_2 = 0.8$ rad, $\varphi_3 = 1.2$ rad, $\varphi_4 = 1$ rad, $l_1 = 5$, $l_2 = 47$, $l_3 = 103$, $l_4 = 205$ and $M = 1024$. Also, $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta$. δ varies in the range (-0.5, 0.5) with an increment of 0.04.

Fig. 1 presents the modulus of the absolute error of the δ_k, $k = 1, 2, 3, 4$, estimates obtained by the proposed method and by IpDFT method [2] as a function of δ.



(a)



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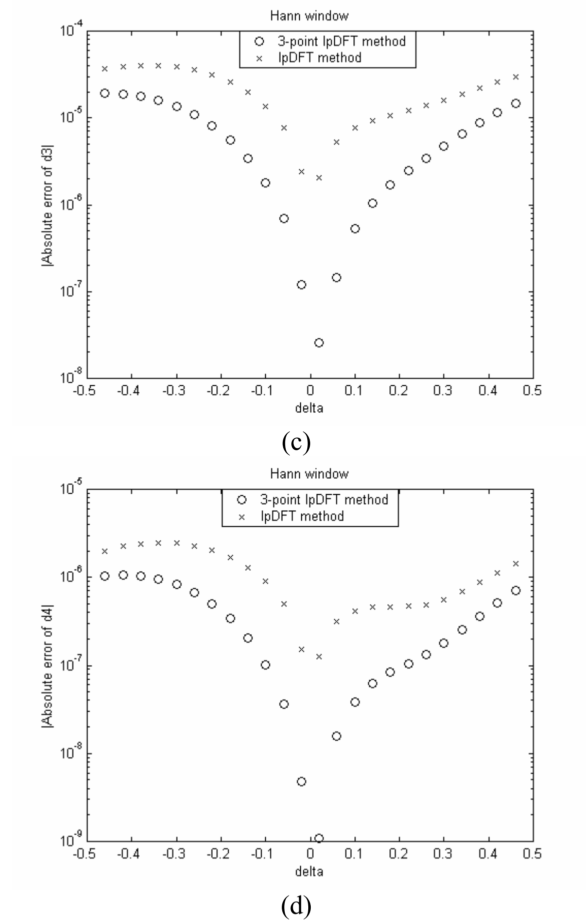


Fig. 1. The modulus of the absolute errors of the δ_k estimates obtained by the proposed method ('o') and by IpDFT method ('x') as a function of δ.

From Fig. 1 it is clearly evident that the three-point IpDFT method leads to more accurate δ_k estimates than the IpDFT method.

In practice the multifrequency signal is affected by quantization noise due to the digitizing process. From this reason the effectiveness of the proposed method has been analysed by simulation in the case when the multifrequency signal (15) is corrupted by quantization noise. Suppose that the signal (15) is applied to an ideal acquisition system with an n-bit analog-to-digital converter (ADC). Thus, the signal (15) is affected only by the quantization noise of the ADC. The ADC resolution, n, varies in the range [6, 24] bits with an increment of 2 bits. The ADC full-scale range is equal to FSR = 5. It is assumed that the quantization noise is uniformly distributed and the quantization errors from sample to sample are statistically independent. It used δ_1 = 0.1, δ_2 = -0.4, δ_3 = 0.3 and δ_4 = -0.25. For each n value 5000 runs are used.

Fig. 2 shows the modulus of the bias of the δ_k estimates as a function of n. δ_k are estimated by the proposed method and by IpDFT method [2].

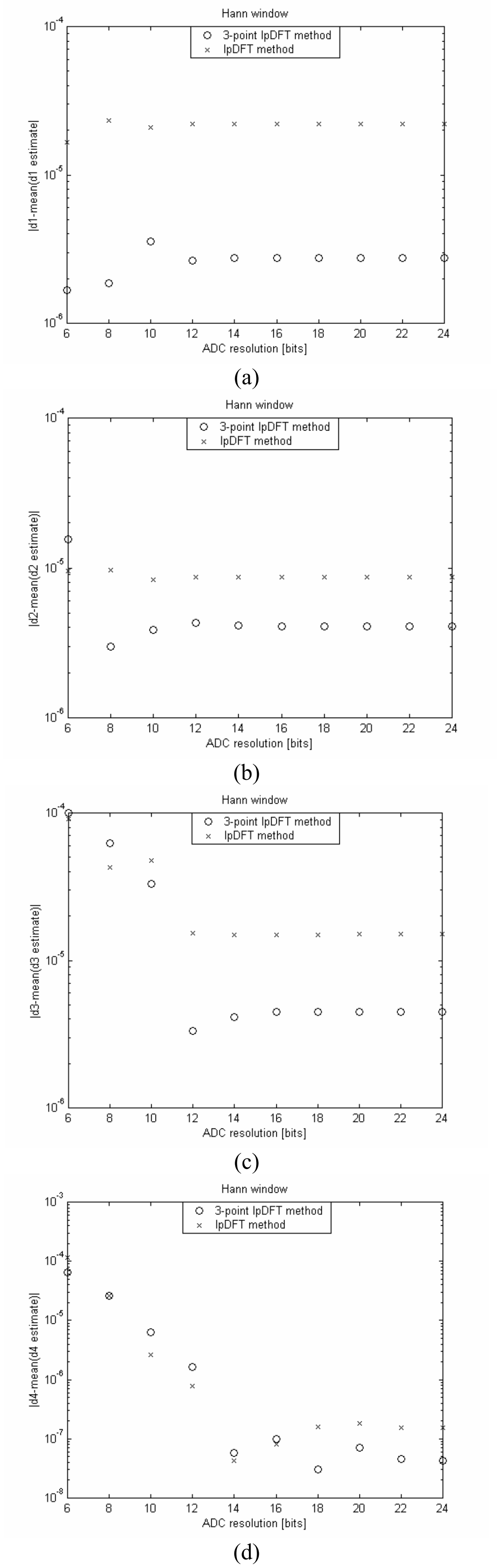


Fig. 2. The modulus of the bias of the δ_k estimates as a function of ADC resolution. δ_k are estimated by the proposed method ('o') and by IpDFT method ('x').

From Fig. 2 it can be observed that for the first three components the proposed method provides more accurate δ_k estimates than the IpDFT method. When l_k becomes relatively large the results obtained by both methods are relatively close due to the fact that the systematic errors of the δ_k estimates become more closely to the quantization noise (the case of the fourth component). When the systematic errors become smaller than the quantization noise then the results obtained by the proposed method practically becomes the same with the ones obtained by IpDFT method.

Due to the quantization error the standard deviation of the δ_k estimates obtained by IpDFT method is computed by [5, eq. (22)]. After some calculus this is given by

- if $-0.5 \le \delta_k < 0$

$$\sigma_{\delta_k} = \frac{(1-\delta_k)(2+\delta_k)\pi\delta_k(1-\delta_k^2)}{3A_k \sin(\pi\delta_k)} \times \sqrt{\frac{3}{M}\left(1-\frac{4}{3}\beta_{1k}+\beta_{1k}^2\right)} \cdot \sigma_q \qquad (16a)$$

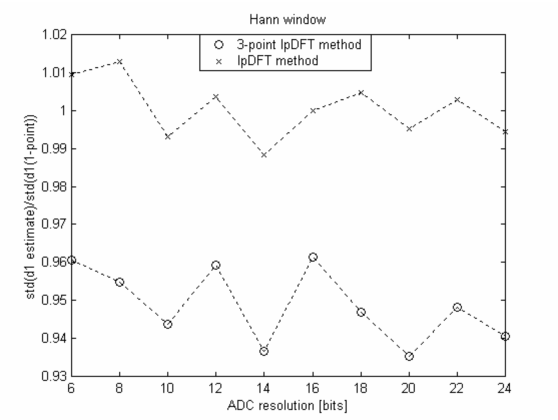
- if $0 \le \delta_k < 0.5$

$$\sigma_{\delta_k} = \frac{\pi\delta_k(2-\delta_k)^2(1-\delta_k^2)}{3A_k \sin(\pi\delta_k)} \times \sqrt{\frac{3}{M}\left(1-\frac{4}{3}\beta_{2k}+\beta_{2k}^2\right)} \cdot \sigma_q \qquad (16b)$$

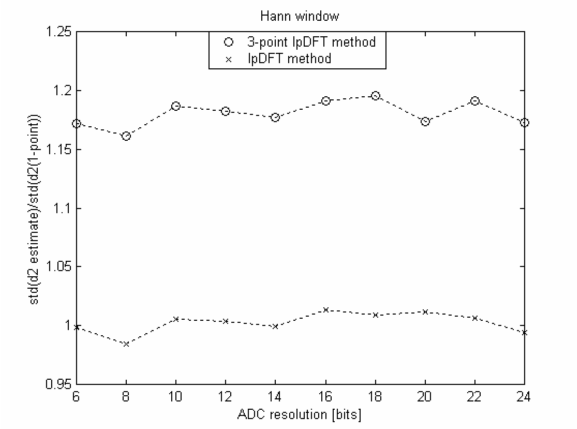
where: σ_q is the quantization noise standard deviation ($\sigma_q = FSR/(2^n\sqrt{12})$);

$$\beta_{1k} = \frac{2+\delta_k}{1-\delta_k} \text{ and } \beta_{2k} = \frac{1+\delta_k}{2-\delta_k}.$$

Fig. 3 shows the rapport between the standard deviation of the δ_k estimates obtained by the three-point IpDFT and IpDFTmethods and the one given by (16) as a function of the ADC resolution.



(a)



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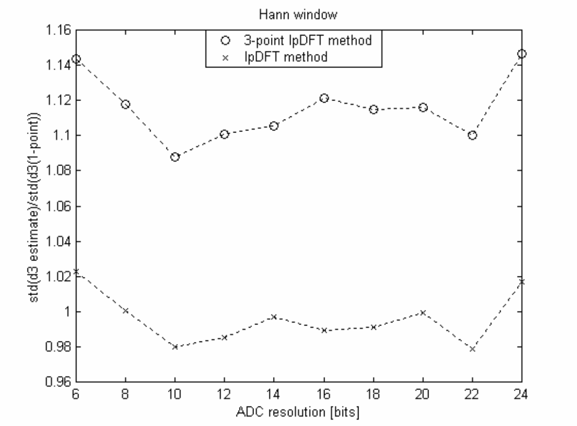
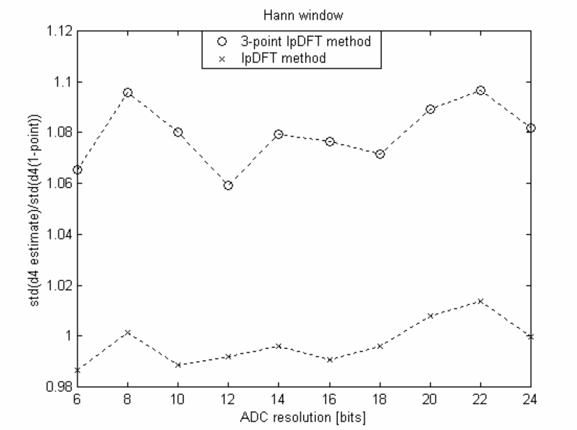


(c)



(d)

Fig. 3. The rapport between the standard deviation of the δ_k estimates obtained by three-point IpDFT and IpDFTmethods and the one given by (16) as a function of the ADC resolution.

From Fig. 3 it follows that the standard deviation of the δ_k estimates obtained by IpDFT method are very closely to the theoretical ones given by (16).

With the exception of the first component, it can be observed that the standard deviations of the δ_k estimates obtained by the proposed method are somewhat higher than the ones obtained by IpDFT method.

IV. CONCLUSION

In this paper a three-point IpDFT method for estimating with high accuracy the frequency of a multifrequency signal component is presented. This method uses the Hann window. Analytical formula for estimating the frequency of a multifrequency signal component by the proposed method is derived.

In the presence of quantization noise for not large l_k values the proposed three-point IpDFT method provides more accurate δ_k estimates than the IpDFT method. When the systematic errors of δ_k estimates obtained by the three-point IpDFT method becomes smaller than the quantization errors then the results obtained by both methods are very closely.

Also, it has been shown that the standard deviations of the δ_k estimates obtained by the proposed three-point IpDFT method are somewhat higher than the ones obtained by IpDFT method.

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