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Chaos in Switching Power Converters

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Abstract - This paper presents an overview of the complex behaviour of the switching power converters. The power electronics circuits, due to their nonlinearity, exhibits a variety of complex behaviour, such as: sudden change of operating regime, subharmonic and chaotic operation, etc. This behaviour can occur when some parameters of the circuit are varied.
Keywords: switching power converters, nonlinear behaviour, chaos

I. INTRODUCTION

Power electronics circuits can be described as piecewise-switched circuits, which assume different topologies at different times. Toggling between these topologies is done in a cyclic manner. The result is nonlinear time-varying operation.

Power electronics circuits, being nonlinear, exhibit a variety of complex behaviour such as sudden change of operating regime, chaotic operation, occasional instability (depending on the circuit parameters), intermittent subharmonic or chaotic operation, etc.

Both the circuit topology and the control method determine the dynamical behaviour of a power electronics circuit. Chaos is a common phenomenon in power converters when they are operated under feedback control. Chaotic systems are sensitively dependent on the initial conditions, which makes long-term prediction of their behaviour impossible.

In analysing power electronics circuits much effort has been spent in developing linear models of dc-dc converters. One of the most popular of these models is the state-space averaging. Although it has many advantages, it is approximate [1]. Also, it doesn't predict some instabilities in the circuit, such as the subharmonic instability associated with the current-mode control [2].

An overview of the nonlinear behaviour in dynamic systems ("chaos theory"), the switching power converters models and their application for nonlinear behaviour analysis, and some techniques used for the study of nonlinear and chaotic behaviour are presented in Section II. The nonlinear behaviour in various switching power converters is examined in Section III. Some conclusions are presented in Section IV.

II. METHODS OF STUDYING NONLINEAR BEHAVIOUR IN SWITCHING POWER CONVERTERS

A. *Nonlinear behaviour in dynamic systems*

Even simple systems can behave in a chaotic fashion. The main cause of this behaviour has been identified as nonlinearity.

Chaos is a particular qualitative behaviour of nonlinear systems, which is characterized by an aperiodic and apparently random trajectory [4].

The behaviour of the dynamical systems varies as a function of time. A dynamical system can be described by the following equation:

$$\frac{dx(t)}{dt} = f(x(t), \mu, t) \quad (1)$$

where x is the state variables vector, and μ is the parameters vector. If f depends on time, the system is called *non-autonomous*, and if f doesn't depend on time, the system is called *autonomous*.

The solution of the system is known as the *trajectory*. The equilibrium solution to which the system converges is called an attracting equilibrium solution (*attractor*). A dynamical system can have multiple equilibrium solutions, depending on the system parameters. When a parameter is varied, the behaviour of the system can suddenly change. This phenomenon is called *bifurcation*. Some commonly observed bifurcations are [2], [5]:

- *saddle-node bifurcation* – it is characterized by a sudden loss of a stable equilibrium solution, when a parameter goes beyond a critical value;
- *transcritical bifurcation* – it is characterized by an exchange of stability status between two equilibrium solutions;
- *supercritical pitchfork bifurcation* – the stable equilibrium solution splits into two stable

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equilibrium solutions, at a critical parameter value;

- *subcritical pitchfork bifurcation* – it is characterized by a sudden explosion of a stable equilibrium solution as a parameter goes beyond a critical value;
- *period-doubling bifurcation* – it is characterized by a sudden doubling of the period of a stable periodic orbit when a certain parameter is varied; this doubling of the period may continue to occur when the parameter is varied in the same direction;
- *Hopf bifurcation* – it is characterized by a sudden expansion of a stable fixed point to a stable limit cycle;
- *border collision* – it is an abrupt change in behaviour when a parameter is varied across the boundary of two structurally different systems. In switching converters it is a result of a change of topological sequence.

B. *Switching power converters modelling*

The nonlinear and time-varying operation of the switching power converters demands nonlinear methods for analysis.

The switching power converters modelling has two basic approaches: averaging approach and discrete-time approach.

Suppose the switching converter toggles between N circuit topologies, d_i being the fraction of the period in which the circuit stays in the i-th topology, A_i and B_i the system matrices, V_g the input voltage and T_s the switching period. Obviously, $d_1 + d_2 + \cdots + d_N = 1$. The state equations for the system are the following:

$$\dot{x} = \begin{cases} A_1 x + B_1 V_g, & t_n \le t < t_n + d_1 T_s \\ A_2 x + B_2 V_g, & t_n + d_1 T_s \le t < t_n + (d_1 + d_2) T_s \\ \ldots & \\ A_N x + B_N V_g, & t_n + (1 - d_N) T_s \le t < t_{n+1} \end{cases} \quad (2)$$

The averaging approach [6] removes the time-varying dependence from the original time-varying model.

The averaged model is the weighted average of all the state equations, written for all possible circuit topologies. The typical form of the averaged model is the following:

$$\frac{dx}{dt} = \left(\sum_{i=1}^{N} d_1 A_i \right) x + \left(\sum_{i=1}^{N} d_1 B_i \right) V_g \quad (3)$$

The control law is given as a set of equations defining d_i. The general form of this set of equations is:

$$\begin{cases} G_1(d_1, d_2, \ldots, V_g, x) = 0 \\ G_2(d_1, d_2, \ldots, V_g, x) = 0 \\ \quad \ldots \end{cases} \quad (4)$$

The averaging approach is widely used and well known, and it is relative easy to derive the continuous averaged equation. Usually, the validity of averaged models is restricted to the low-frequency range, up to an order of magnitude below the switching frequency. For this reason, averaged models become inadequate when the aim is to explore nonlinear phenomena that may appear across a wide spectrum of frequencies. Nevertheless, averaging techniques can be useful to analyze low-frequency bifurcation phenomena.

Another modelling approach is the discrete-time iterative approach. Its aim is to derive an iterative function that expresses the state variables at one sampling instant in terms of those at an earlier sampling instant:

$$x_{n+1} = f(x_n, d, V_g) \quad (5)$$

where x_n is the state vector at $t = nT_s$, d is the vector of the duty cycles: $d = [d_1\ d_2\ \ldots\ d_N]^T$. Eqn (5), the discrete-time state equation, assumes that the sampling period is equal to the switching period. Therefore, the model can be used up to the switching frequency. Since most power electronics circuits are non-autonomous systems driven by fixed-period clock signals, the study of the dynamics can be effectively carried out using appropriate discrete-time maps. The disadvantage of the model is that the derivation of the iterative map is more complicated compared to the continuous-time averaged equation.

C. *Analysis of standard bifurcations*

The analysis begins with the system model.

If the averaged model is used, first the set of continuous averaged state equations are derived. Then, the eigenvalues (characteristic multipliers) of the Jacobian, $J(X_Q)$ and are found, using the following equation:

$$\det[\lambda I - J(X_Q)] = 0 \quad (6)$$

The next step is to identify the condition for the eigenvalue(s) to move across the imaginary axis in the complex plane (as for instance, a pair of complex eigenvalues moving across the imaginary axis implies a Hopf bifurcation).

If the discrete time approach is used, first is derived the discrete-time state equation (also called iterative map, iterative function or Poincaré map). Next, the Jacobian, $J(X_Q)$ is examined to find eigenvalues. Then, the condition for the eigenvalue(s) to move out the unit circle in the complex plane is identified.

The discrete-time and averaged models treat the duty cycle as an input. In practice, the duty cycle is controlled through some feedback mechanisms. Thus, to complete the model, we need to state the control law. For instance, in the usual pulse-width modulation control, a control signal (deriving from the state variables) and a ramp signal are compared, their intersection defining the switching instant. Thus, the control law can be:

$$V_{ramp}(dT_s) = v_{con}(x(dT_s)) \tag{7}$$

where $V_{ramp}(t)$ is a ramp voltage and v_{con} is the control signal. From this equation, we can find d for each switching period. For the voltage-mode control, considering a proportional feedback, the control law is the following:

$$d_n = H\left(D - \kappa\left(v_o - V_{ref}\right)\right) \tag{8}$$

where D is the steady-state duty cycle, κ is the small-signal feedback gain, V_{ref} is the reference output voltage, and H limits the range of the duty cycle between 0 and 1:

$$H(x) = \begin{cases} 0, & for \quad x < 0 \\ 1, & for \quad x > 1 \\ x, & for\, 0 \le x \le 1 \end{cases} \tag{9}$$

For the current-mode control, the control law is given by:

$$i_{ref} = I_{ref} - \kappa(v_o(t) - V_{ref}) \tag{10}$$

where I_{ref} is the steady-state reference current.

D. *Techniques for studying nonlinear and chaotic behaviour*

Poincaré sections – are graphical representations of the behaviour of a high-order system. They are planes that intersects the trajectory of the system. If the Poincaré section contains a finite number of points, the steady-state operation is periodic. If the Poincaré section contains a closed loop, the operation is quasi-periodic. If the Poincaré section is irregular, the operation is chaotic.

Bifurcation diagrams – are graphical representations of the behaviour exhibited by a system when some parameters are varied.

Lyapunov exponents – they measure the exponential convergence or divergence of neighbouring orbits of a dynamical system. Considering two trajectories that initially are separated by a distance ε_0, if this distance increases or decreases exponentially in time, it can be expressed as:

$$\varepsilon(t) = \varepsilon_0 e^{\lambda t} \tag{11}$$

If $\lambda > 0$, the two trajectories diverge exponentially in time and the behaviour of the system is chaotic.

III. NONLINEAR BEHAVIOUR IN SWITCHING POWER CONVERTERS

Switching power converters, due to their nonlinearity exhibits a variety of complex and chaotic behaviour. The behaviour of a dc-dc converter is greatly influenced by the operating mode and the control technique. Most dc-dc converters are designed to deliver a regulated output voltage. The control of dc-dc converters usually takes on two approaches: *voltage feedback control (voltage-mode control)* and *current-programmed control (current-mode control)*.

In voltage feedback control, the output voltage is compared with a reference to generate a control signal which drives the pulse-width modulator.

For current-programmed control, an inner current loop is used in addition to the voltage feedback loop, to force the peak inductor current to follow a reference signal which is derived from the output voltage feedback loop.

From what has been reported so far in the literature, the following observations regarding the nonlinear behaviour of the switching power converters can be made:

- converters under fixed-frequency current-mode control generally lose stability via a period-doubling type of bifurcation;
- in switching power converters border collision can occur due to a change of operating mode or due to saturating nonlinearity– caused by the inherent limitation of the range of some control parameters (as for instance, as a result of saturating the duty cycle [2]);
- voltage-mode controlled BUCK converters typically exhibit period-doubling bifurcations [12, 13], whereas BOOST converters typically exhibit Hopf bifurcation [10, 14];
- period-doubling is common in BUCK or BOOST converters operating in Discontinuous Inductor Current Mode (DICM) [15, 16] and current-mode controlled converters [11, 17].

Further on, we analyse the nonlinear behaviour of some switching power systems.

A. *BUCK converter, voltage-mode control, CCM operation*

For the BUCK converter, shown in Fig. 1, the state-equations when the converter is operating in Continuous Conduction Mode (CCM), are the following:

$$\begin{aligned} \dot{x} &= A_1 x + B_1 V_g, \quad t_n \le t < t_n + dt \\ \dot{x} &= A_2 x + B_2 V_g, \quad t_n + dt \le t < t_{n+1} \end{aligned} \tag{12}$$

where x is the state vector $[v_o \; i_L]^T$, d is the duty cycle.

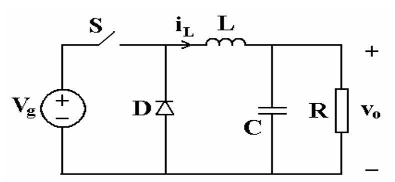

Fig. 1. The BUCK converter.

By solving the state equations, the discrete-time equation can be obtained:

where:

$$f(x,d)=\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} x+\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} V_g \tag{14}$$

The approximate expressions for *f(x,d),*determined in [2] are given by:

$$f_{11}=1-\frac{T_s}{\tau_C}+\frac{T_s^2}{2\tau_C^2}-\frac{T_s^2}{2\tau_C\tau_L} \tag{15}$$

$$f_{12}=\frac{RT_s}{\tau_C}-\frac{RT_s^2}{2\tau_C^2} \tag{16}$$

$$f_{21}=-\frac{T_s}{R\tau_L}+\frac{T_s^2}{2R\tau_C\tau_L} \tag{17}$$

$$f_{22}=1-\frac{T_s^2}{2\tau_C\tau_L} \tag{18}$$

$$g_1=\left(1-\frac{d}{2}\right)\frac{dT_s^2}{\tau_C\tau_L} \tag{19}$$

$$g_2=\frac{dT_s}{R\tau_L} \tag{20}$$

where $\tau_C=CR,\ \tau_L=L/R$.

For voltage-mode controlled BUCK converter, operating in CCM [2], in order to investigate the bifurcation phenomena, the following parameters are used: V_g = 22-33V, L = 20mH, f_s = 2.5kHz, R = 22Ω, C = 47μF, V_{ref} = 11V.

In Fig. 2, 3, 4 are presented the output voltage and inductor current waveforms and the phase portraits in various operating regimes: fundamental periodic operation (period-1), period-2 subharmonic operation and chaos, as a result of the border collision, obtained by computer simulation, using CASPOC.

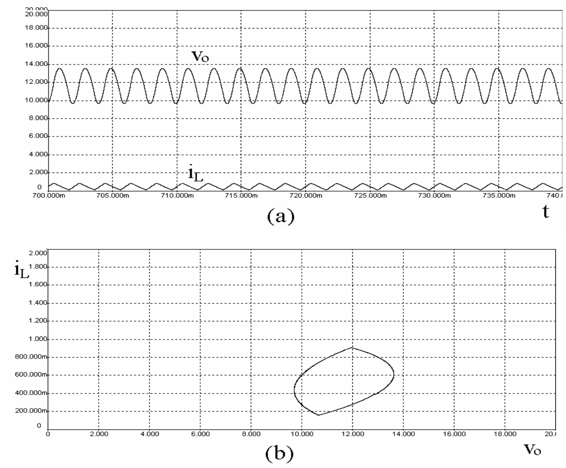

Fig. 2. (a) Fundamental waveforms (simulation results) for BUCK converter operating in CCM, V_g=23V; (b) phase portrait.

B. *BUCK converter, voltage-mode control, DICM operation*

In a similar way to CCM, the state equation for DICM can be derived:

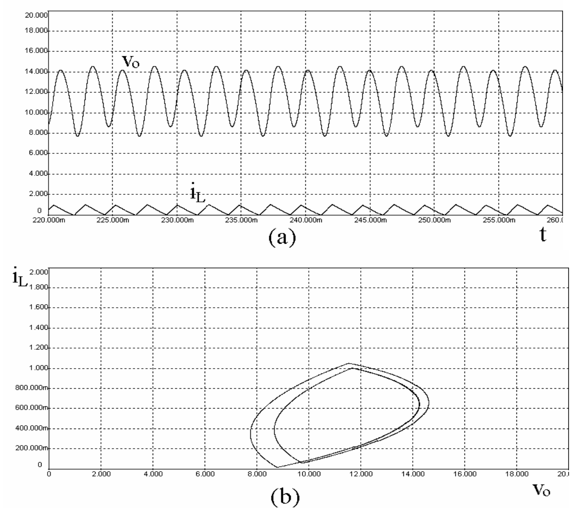

Fig. 3. (a) Period-2 subharmonic waveforms (simulation results) for BUCK converter operating in CCM, V_g=28V; (b) phase portrait.

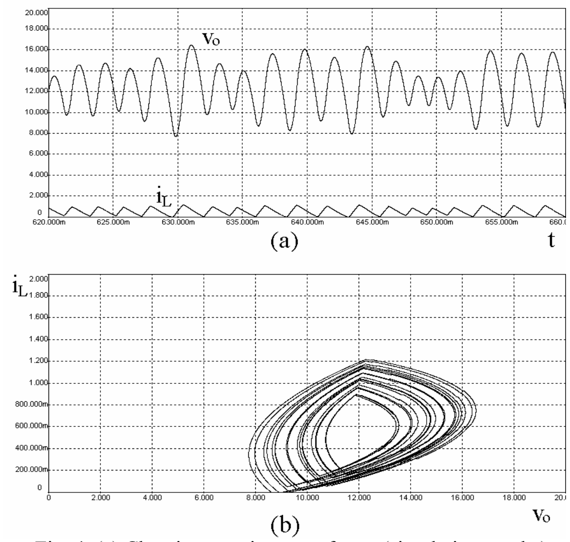

Fig. 4. (a) Chaotic operation waveforms (simulation results) for BUCK converter operating in CCM, V_g=33V; (b) phase portrait.

$$v_o(t_{n+1})=f(v_o(t_n),d_n)=\alpha v_o(t_n)+\frac{d_n^2 V_g(V_g-v_o(t_n))}{v_o(t_n)} \tag{21}$$

where:

$$\alpha=1-\frac{T_s}{\tau_C}+\frac{T_s^2}{2\tau_C^2} \tag{22}$$

$$\beta=\frac{T_s^2}{2LC} \tag{23}$$

For voltage-mode controlled BUCK converter, operating in DICM, in order to investigate the bifurcation phenomena, the following component values are used: V_g = 33V, L = 194μH, f_s = 3kHz, R = 12.5Ω, C = 222μF, V_{ref} = 25V, D = 0.47. Assuming that in the neighbourhood of the steady-state the duty cycle does not saturate, the characteristic multiplier can be computed from Eqn. (21):

$$\lambda = \left.\frac{\partial f(v_o)}{\partial v_o}\right|_{v_0=V_o} = \alpha - \frac{\beta V_g D[2\kappa V_o(V_g - V_o) + DV_g]}{V_o^2} \tag{24}$$

The system is fundamentally stable if $|\lambda| < 1$. The critical value of the small-signal feedback gain, κ, can be found by setting the characteristic multiplier to -1:

$$\kappa_c = \frac{(1+\alpha)V_o^2 - \beta V_g^2 D^2}{2\beta V_g D V_o (V_g - V_o)} \tag{25}$$

For the studied circuit, κ_c=0.115.

In Fig. 5 and 6 are presented the waveforms and the phase portraits in two operating regimes: fundamental periodic operation (period-1) and chaos, obtained by computer simulation.

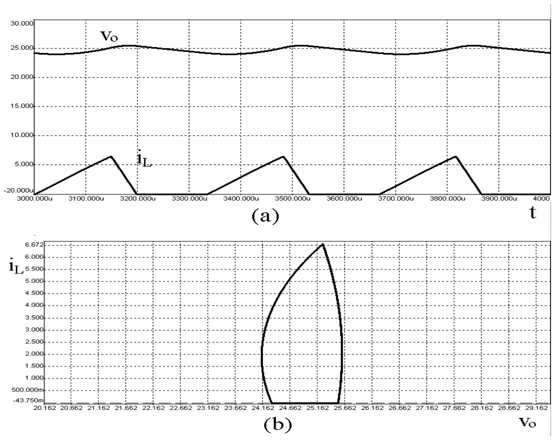

Fig. 5. (a) Fundamental waveforms (simulation results) for BUCK converter operating in DICM ,κ=0.1;
(b) phase portrait.

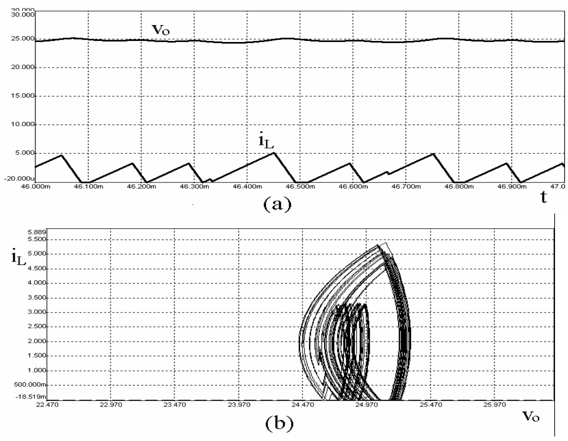

Fig. 6. (a) Chaotic operation waveforms (simulation results) for BUCK converter operating in DICM, κ=0.185;
(b) phase portrait.

In Fig. 7 the analytical iterative map, given by Eqn. (21) is used to show the bifurcations of the converter. It can be observed that the system looses stability by period-doubling, and becomes chaotic when κ becomes larger that about 0.17.

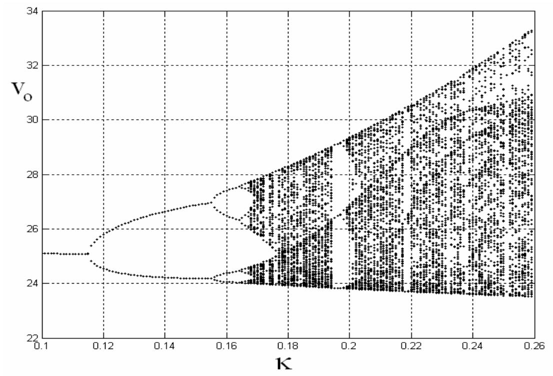

Fig. 7. Bifurcation diagram of the voltage-mode controlled BUCK converter operating in DICM.

The voltage-mode controlled BOOST converter, operating in DICM presents the same nonlinear behaviour [15].

C. *BUCK converter, current-mode control*

In this case, the control law is given by Eqn. (10). For current-mode controlled BUCK converter, in order to investigate the bifurcation phenomena, the following parameters are used: V_g = 5V, L = 2mH, f_s = 10kHz, R = 40Ω, C = 34μF, V_{ref} = 1.89V, I_{ref} = 0.2185A. In Fig. 8, 9, 10 are presented: the inductor current waveforms and the phase portraits in various operating regimes: fundamental periodic operation, period-2 subharmonic operation and chaos, obtained by computer simulation.

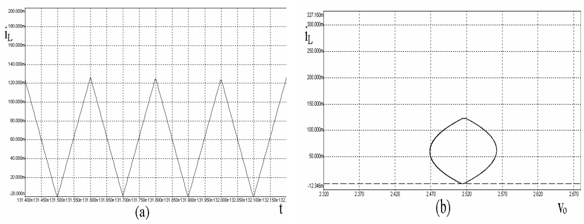

Fig. 8. (a) Fundamental waveform (simulation results) for current-mode controlled BUCK converter, κ=0.15; (b) phase portrait.

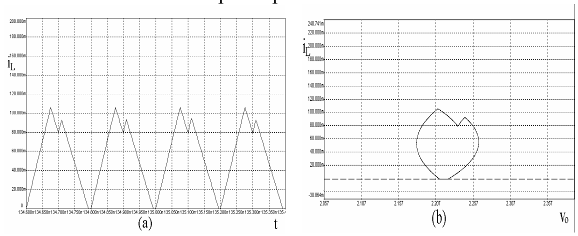

Fig. 9. (a) Period-2 subharmonic waveform (simulation results) for current-mode BUCK converter , κ=0.35;
(b) phase portrait.

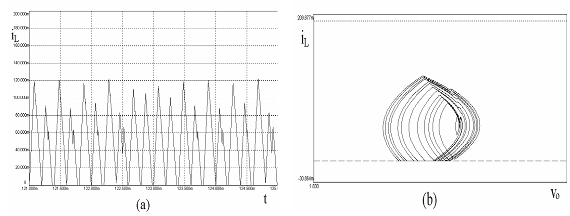

Fig. 10. (a) Chaos operation waveform (simulation results) for current-mode BUCK converter , κ=1.15;
(b) phase portrait.

It can be observed that the circuit goes through 1-period operation, 2-period operation, and eventually exhibits chaos. A similar behaviour can be observed at the BOOST converter [2], [11].

D. *ĆUK converter, Discontinuous Capacitor Voltage Mode (DCVM) operation, voltage-mode control*

In this paper, the ĆUK converter (Fig. 11), operating in DCVM is studied. In order to investigate the occurence of the Hopf bifurcation, the following parameters are used: V_g = 15V, L_1=L_2 = 2.4mH, R = 10Ω, C_1=56.8μF C_2 = 47μF, f_s=20kHz.

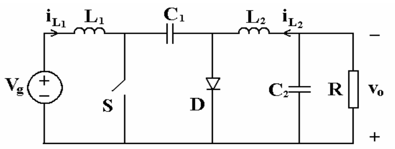

Fig. 11. The ĆUK converter.

The control law is given by (8). For this circuit parameters it is found that the critical value of the feedback gain, κ, is 0.816.

Computer simulations of the circuits show the bifurcation from fixed point (Fig. 12,a), through limit cycle (Fig. 12,b) and eventually to chaos (Fig. 12,c).

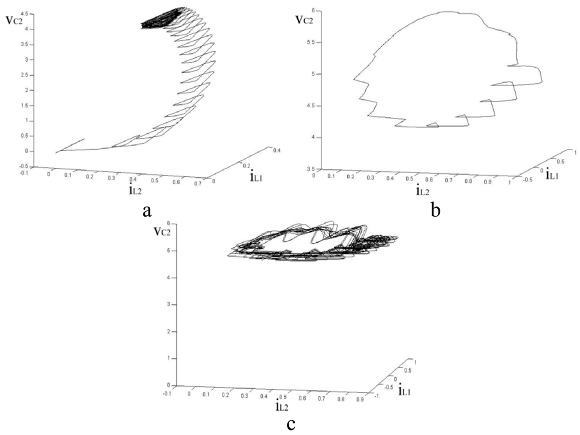

Fig. 12. The 3-d plots of the local trajectory for the DCVM ĆUK converter: a- the stable local trajectory for κ=0.1; b- the limit cycle (κ=0.816); c-chaotic orbit (κ=10).

In [2], [10] the Hopf bifurcation in the free-running ĆUK converter, operating in CCM, the boost converter with PWM voltage-mode control, and parallel boost converters is studied.

IV. CONCLUSIONS

The power electronics circuits, due to their nonlinearity, exhibits a variety of complex behaviour, such as: sudden change of operating regime, subharmonic and chaotic operation, etc. This behaviour can occur when some parameters of the circuit are varied. There are two reasons for studying nonlinear dynamics in the context of power electronics: to understand better the nonlinear behaviour of the power converters, and thereby avoid undesirable effects, or to deliberately use these effects, as shown in [19].

The nonlinear behaviour of some switching power converters is analysed in this paper.

REFERENCES

[1] P. T. Krein, J. Bentsman, R. M. Bass, and B. L. Lesieutre, “On the use of averaging for the analysis of power electronic systems”, *IEEE Transactions on Power Electronics*, vol. 5, no. 2, pp. 192-192, Apr. 1990.

[2] C. K. Tse, *Complex Behaviour of Switching Power Converters*, CRC Press LLC, 2004.

[3] D. C. Hamill, J. H. B. Deane, and D. J. Jefferies, “Modelling of Chaotic DC-DC Converters by Iterated Nonlinear Mappings”, *IEEE Transactions on Power Electronics*, vol. 7, no. 1, January 1992.

[4] H. Peitgen, H. Jugens and D. Saupe, *Chaos and Fractals: New Frontiers of Science*, Berlin: Springer-Verlag, 1992.

[5] H. E. Nusse, E. Ott, and J. A. Yorke, “Border-collision bifurcations: an explanation for observed bifurcation phenomena”, *Physical Review E,* vol. 49, pp. 1073-1076.

[6] R. D. Middlebrook and S. Ćuk, “A general unified approach to modelling switching-converter power stages”, *IEEE Power Electronics Specialists Conference Record,* pp. 18-34, 1976.

[7] D. C. Hamill, “Power electronics: a field rich in nonlinear dynamics”, *Proceedings of International Specialists' Workshop on Nonlinear Dynamics of Electronic Systems*, pp. 165–177, 1995.

[8] D. Dai, C. K. Tse, “Symbolic Analysis of Bifurcation in Switching Power Converters: A Novel Method for Detecting Border Collision”, *2004 RISP International Workshop on Nonlinear Circuit and Signal Processing (NCSP'04),* Hawaii, USA, Mar. 5-7, 2004.

[9] Y. Ma, H. Kawakami, C. K. Tse, “Bifurcation Analysis of Switched Dynamical Systems With Periodically Moving Borders”, *IEEE Transactions on Circuits and Systems-I: Regular Papers,* vol. 51, no. 6, June 2004.

[10] A. El Aroudi, “Hopf bifurcation and chaos from torus breakdown in a pwm voltage-controlled dc/dc boost converters”, *IEEE Transactions on Circuits and Systems Part I*, vol. 46, no. 11, pp. 1374–1382, 1999.

[11] W. C. Y. Chan and C. K. Tse, “Study of bifurcation in current-programmed dc/dc boost converters: from quasiperiodicity to period-doubling”, *IEEE Transactions on Circuits and Systems Part I,* vol. 44, no. 12, pp. 1129–1142, 1997.

[12] K. Chakrabarty, G. Podder and S. Banerjee, “Bifurcation behaviour of buck converter”, *IEEE Transactions on Power Electronics*, vol. 11, no. 3, pp. 439–447, 1995.

[13] D. C. Hamill, J. H. B. Deane and D. J. Jefferies, “Modeling of chaotic dc/dc converters by iterative nonlinear mappings”, *IEEE Transactions on Power Electronics,* vol. 7, no. 1, pp. 25–36, 1992.

[14] H. H. C. Iu and C. K. Tse, “Bifurcation in parallel-connected boost dc/dc converters”, *Proceedings of IEEE International Symposium on Circuits and Systems,* pp. II473–II476, 2000.

[15] C. K. Tse, “Flip bifurcation and chaos in three-state boost switching regulators”, *IEEE Transactions on Circuits and Systems Part I,* vol. 41, no. 1, pp. 16–23, 1994.

[16] C. K. Tse, “Chaos from a buck regulator operating in discontinuous conduction mode”, *International Journal of Circuit Theory and Applications,* vol. 22, pp. 263–278, 1994.

[17] C. K. Tse and W. C. Y. Chan, “Chaos from a current-programmed Ćuk converter”, *International Journal of Circuit Theory and Applications,*vol. 23, no 3, pp. 217–225, 1995.

[18] W. C. Y. Chan and C. K. Tse, “Bifurcations in current-programmed dc/dc buck switching regulators-conjecturing a universal bifurcation path”, *International Journal of Circuit Theory and Applications, 26,* pp. 127–145, 1998.

[19] J.H.B. Deane and D.C. Hamill, “Improvement of power supply EMC by chaos”, *Electronics Letters*, vol. 32, no. 12, p. 1045, June 1996.