Buletinul Ştiinţific al Universităţii "Politehnica" din Timişoara

Seria ELECTRONICĂ şi TELECOMUNICAŢII
TRANSACTIONS on ELECTRONICS and COMMUNICATIONS

Tom 51(65), Fascicola 2, 2006

ECG Signal Denoising in the Diversity Enhanced Wavelet Domain

Marius Oltean[1], Victor Adafinoaiei [2]

Abstract – The paper presents a denoising algorithm using which is particularly suited to ECG signals. The main stage of this algorithm consists in a MAP filtering in wavelet domain. Its effectiveness relies on the diversity enhancement of the signal to be processed and on realistic a-priori assumptions regarding statistical properties of the wavelet coefficients. Tests made on a big number ECG signals, in realistic conditions, showed very promising results. The noise is removed, while the useful waveforms are preserved.
Keywords: denoising, wavelet, ECG , MAP

I. INTRODUCTION

The clinical electrocardiogram (ECG) records the changing potentials of the electrical field generated by the heart. Electrocardiography can be used, within limits, to identify anatomical, metabolic, ionic and hemodynamic changes. Automatic ECG signal processing aims the detection and even the prevention of cardiac illness and can be very helpful for the cardiologists. Unfortunately, ECG signal acquisition process is subjected to various disturbing perturbations like power-line interferences, electromyogram noise caused by muscle activity, motion artifacts and baseline drift due to the respiration mechanism. All these unwanted phenomena make from the automatic interpretation of the signal a difficult and sometimes even an impossible task. In these conditions, a pre-treatment of the signal is highly desirable for removing such interferences. This procedure will be next referred to as denoising.

The term was introduced by Donoho [1] in relation with the wavelet transform (WT). WT has been extensively used in the signal processing community in order to highlight informative representations of non-stationary signals. WT is able to simultaneously provide time and frequency information and offers good temporal localization for high frequencies and high–frequency resolution for low frequencies. ECG record is a non-stationary signal, so WT-based denoising particularly matches to it. The architecture of a wavelet-based denoising system relies on WT ability to concentrate the useful signal energy into a small number of wavelet coefficients. The algorithm introduced by Donoho uses discrete wavelet transform (DWT) and it has three steps:

1. DWT is applied on the noisy signal;
2. Wavelet coefficients are filtered (procedure which is sometimes referred to as “shrinkage”, or “thresholding”). In general, some of these coefficients are put to 0, since they don’t contain useful information;
3. Remaining coefficients are back-converted in time domain to estimate the useful signal.

Generally, the results are highly dependent on the wavelet mother used (stages 1 and 3) and on the filtering procedure chosen (stage 2). Some modern wavelet denoising techniques implement a MAP filtering in the stage 2 of the algorithm, taking into account the statistical properties of the wavelet coefficients. Such a method, which is used for processing the ECG signal in noisy conditions [2], adapts the wavelet domain empirical Wiener filter presented in [3] to the particular case of ECG signals. The statistical properties of the wavelet coefficients are estimated through a pilot signal. The pilot is obtained by applying the classical Donoho’s algorithm on the input noisy signal. Next, a MAP filtering in wavelet domain is performed, using the properties estimated through the pilot. The wavelet basis used in the two stages (pilot estimation and MAP filtering) are different. Using a wavelet basis function with compact temporal support in the first stage allows for an accurate preservation of the areas around the QRS complex [2]. On the other hand, the use of wavelets with good frequency localization in the second stage of the algorithm refines the shapes of P and T waves. Note that Wiener filter could be regarded as a particular case of a MAP filter. The analytical solution required for implementing this kind of filter uses the hypothesis that both useful and noise samples (wavelet coefficients when the filter is applied in WT domain) have Gaussian probability density function (pdf). The two most important features of such a filtering technique are: realistic a-priori assumptions regarding statistical properties of both signal and noise components and a good

[1] Teach Assistant,[2] Student: Facultatea de Electronică şi Telecomunicaţii, Departamentul Comunicaţii Bd. V. Pârvan Nr. 2, 300223 Timişoara, e-mail marius.oltean@etc.upt.ro

estimation of the parameters that describe these properties.

An improved method to estimate the statistical parameters of the wavelet coefficients is proposed in this paper. This method relies on the diversity enhancement of the signal to be processed. On the other hand, realistic a-priori assumptions were made regarding the statistical properties of the wavelet coefficients. These assumptions are well adapted to the characteristic shape of ECG signal.

In ECG denoising, exact preservation of the useful waveforms is critical. However, distortions are sometimes introduced in the useful signal by DWT based denoising. In our algorithm, these distortions are mitigated by the use of a redundant WT, which provides translation invariance. On the other hand, the distortions can be controlled by a proper selection of the mother wavelet function and its corresponding scaling function. In the present study, after averaging ten results that we got by using different wavelet mothers, SNR improvement was obtained. In the same time, we show that there isn't a wavelet mother that offers the best results in all situations.

The theoretical background of our algorithm especially considers the suppression of wide band EMG noise, but good practical results are provided for the power-line interference too.

In section II, the proposed denoising algorithm is presented. Next, simulation results are shown. Section IV contains a few concluding ideas and draws future possible directions to continue our work on this subject.

II. METHOD

The architecture of the proposed denoising system is presented in fig. 1.

To the input we get the useful signal (s) additively perturbed by a Gaussian colored noise (p):

$$x = s + p \qquad (1)$$

The denoising procedure is composed of two stages, presented below.

Stage1: Pilot signal and noise estimation

The goal of this stage is to provide a reliable estimation of both "clean" signal and noise statistical parameters.

In this purpose, the classical denoising method proposed by Donoho [1] is applied in the wavelet domain W1. Thus, the signal is converted in the wavelet domain, the resulting wavelet coefficients are shrinked ("Sh" block, fig. 1) and then back-converted in time domain.

As illustrated in [3], the use of a wavelet mother with compact temporal support is recommended in this stage. This choice mitigates pseudo-Gibbs phenomenon effects (ripples around discontinuities), usually associated with the shrinkage of the DWT coefficients. Thus, a good preservation of the zones around QRS is provided in this stage.

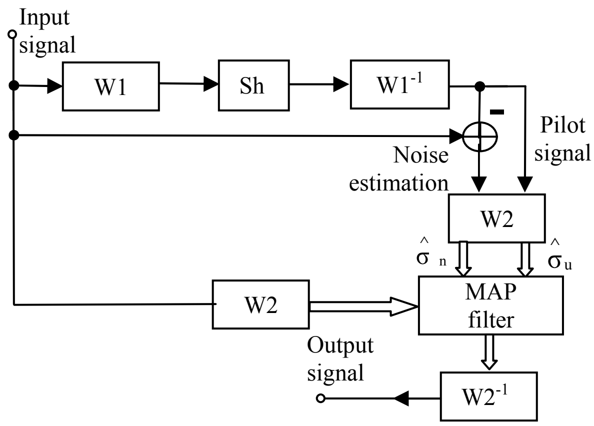

Fig.1 : Architecture of the denoising system.

The estimation of the pilot signal plays a double role. Besides the estimation of the "clean" signal, an estimation of the noise can be obtained as the difference between the noisy "observed" signal and the pilot signal. This operation takes into account the additive nature of the noise, illustrated by equation 1.

Thus, the first stage of the algorithm provides two time-domain signals: a pilot signal (estimating the useful signal) and a "purely" noise signal estimation respectively.

Stage 2: MAP filtering in the diversity-enhanced wavelet domain W2

In this stage, an empirical MAP filtering in the wavelet domain W2 is implemented. In order to provide robustness and superior performance to our algorithm, we made realistic a-priori assumptions regarding pdf of the useful and noise coefficients. In the same time, the statistical parameters estimation is improved by the diversity-enhancement of the signal to be processed. The diversity enhancement is obtained in the wavelet domain, by combining two redundant wavelet transforms, each of them providing several versions of the signal to be processed. The sources of diversity are the type of wavelet mother used in the computation of the discrete wavelet transform (DWT) [4] and the circular translation of the signal samples respectively [5]. In the first case we consider L_1 different wavelet mothers. In the second one, L_2 circular translations of the signal samples are used, but only one wavelet mother (chosen from the L_1 options). The two transforms are known as diversity-enhanced DWT (DEDWT) [4] and translation invariant DWT (TIDWT) [5] respectively. These transforms are combined in the following manner: L_1 versions of TIDWT are performed, each of them corresponding to a different wavelet mother. A new transform is obtained, called TIDWTED (Translation Invariant Wavelet Transform with Enhanced Diversity). Its redundancy is $L=L_1 x L_2$. In our denoising system (see fig. 1), this transform is denoted by W2. Thus, to the output of the W2 block, we get L sequences of discrete wavelet coefficients, as follows:

$$^{l}w = {}^{l}u + {}^{l}n, \quad l = 1,\ldots, L \qquad (2)$$

${}^{l}u$ and ${}^{l}n$ denoting the useful and the noise coefficients respectively, for the l-th set of wavelet coefficients.

Using Bayesian rules, the MAP estimation of ${}^{l}u$ can be computed as:

$$\begin{aligned}{}^{l}\hat{u}\,({}^{l}w) &= \arg\max_{{}^{l}u}\left(\log(p_{w/u}({}^{l}w/{}^{l}u)\cdot p_u({}^{l}u))\right)= \\ &= \arg\max_{{}^{l}u}\left(\log(p_n({}^{l}w-{}^{l}u))+\log(p_u({}^{l}u))\right)\end{aligned} \quad (3)$$

In the following, without loss of generality, we can consider for the noise coefficients a Gaussian distribution, with zero mean and variance σ_n^2:

$$p_n(n)=\frac{1}{\sqrt{2\pi}\sigma_n}\exp\left(-\frac{n^2}{2\sigma_n^2}\right) \quad (4)$$

For the useful signal coefficients pdf (p_u), a Laplacian distribution seems to be well suited to the characteristic shape of the ECG signal. This supposition is supported by empirical work on large ECG databases [6]. In fact, the wavelet transform of an ECG signal consists into a small number of high value wavelet coefficients (especially marking the limits of the electrical activity zones) and a large number of small value coefficients (for the slow-evolution portions of the ECG). A heavy-tailed distribution for these coefficients seems therefore far more realistic than a Gaussian-one, and the particular case of a Laplacian probability density function (pdf) becomes attractive by its computational tractability. Consequently, we take:

$$p_u(u)=\frac{1}{\sqrt{2}\sigma_u}\exp\left(-\frac{\sqrt{2}|u|}{\sigma_u}\right) \quad (5)$$

Under the considered assumptions, the solution of (3) is [6,7]:

$${}^{l}\hat{u}=\begin{cases}{}^{l}w-{}^{l}T, & \text{if } \left|{}^{l}w\right|>{}^{l}T \\ 0, & \text{otherwise}\end{cases}, \quad l=1,\dots,L \quad (6)$$

This solution represents a softthresholding filtering of each of l sequences of noisy observations with the optimal threshold value ${}^{l}T$. This value is computed using the estimated standard deviations of the pilot and noise coefficients (see stage 1):

$${}^{l}T(j,k)=\frac{\sqrt{2}\,{}^{l}\hat{\sigma}_n^2(j)}{{}^{l}\hat{\sigma}_u(j,k)} \quad (7)$$

Note that the threshold value is individually estimated for each coefficient w(j,k), positioned on the j-th decomposition scale and having the index k within the scale. This is highly recommendable, since $\hat{\sigma}_u$ (estimated standard deviation of the useful coefficients) must be performed locally, in order to accurately track the ruptures that exist in the signal (e.g. the QRS complex). This parameter is separately estimated for each coefficient, using a sliding window:

$${}^{l}\hat{\sigma}_u(j,k)=\frac{\sqrt{\sum_i|\xi(j,i)|^2}}{\upsilon}, \quad i=k-\frac{\upsilon-1}{2},\dots,k+\frac{\upsilon-1}{2} \quad (8)$$

where $\xi(j,i)$ represents the wavelet coefficient of the pilot signal, j standing for the decomposition scale and i for the position within the scale. ν is the length of the sliding window. Experimental work showed that the value ν=1 provides comparative results with higher window lengths (ν=3 or 5), so this value is chosen. On the other hand, the noise variance is separately estimated at each decomposition level j, using the wavelet coefficients of the purely noise signal at that level. This approach takes into account the fact that, generally, the noise that affects an ECG signal is not white, so its variance changes within scales (different frequency subbands).

Finally, useful signal is estimated by averaging all L versions of the estimated signal. This implies L_2 un-shifting operations, and then an averaging-over-shifts, performed by the Inverse Translation Invariant DWT (ITIDWT) [5]. Remember that we applied this transform for L_1 different wavelet mothers. The final result is obtained by averaging the L_2 variants of the denoised signal. As observed, the wavelet transform used (TIDWTED) is double redundant. The translation invariance is offered by averaging over the circular shifts. This mitigates the problem of peudo-Gibbs oscillations around fast transition portions of the signal (QRS area). The system performance is not sensitive to the wavelet mother chosen, since several basis functions are simultaneously used. Both transforms improve the SNR performance, by the averaging operation.

III. RESULTS

Several simulation sets were performed on real ECG signals, in order to demonstrate the performance of the proposed method.

3.1 General simulation parameters

ECG test signals were chosen from CHU Brest database. The sampling frequency of these signals is of 1000 Hz, with a resolution of 16 bits/sample. In order to obtain the pilot estimation (stage I), we shrinked the Haar coefficients of the noisy signal, with the threshold value $T(j)=s(j)\sqrt{2\log M}$ [8], where s(j) represents the standard deviation of the noisy wavelet coefficients at the decomposition level j and M is the length of the data block, namely M=4096 samples. For the second stage of the algorithm, we have chosen for DEDWT implementation L_1=10 different wavelet mothers with good frequency localization, from Daubechies, Coiflet and Symmlet families. In the case of TIDWT, we used the "fully" TIDWT [5], which averages over all circular shifts of the signal. That is, in this case, we get L_2=4096. Note that this transform can be calculated rapidly, in M logM time, despite appearances. This way, the redundancy factor is $L=L_1 x L_2$=40960.

3.2 Simulation sets

In order to correctly evaluate the method's performance, several types of simulations were performed. Thus, SNR improvement was estimated. On the other hand, we evaluated the denoising effects on the next stage of the automatic processing chain, namely signal segmentation.

3.2.1 SNR improvement

SNR improvement represents a classical measure of denoising quality. In order to compute this measure, the clean signal must be a-priori known. In this context, we chose 5 "clean" ECG test signals of 60 seconds each. Artificially generated noise was added to this signal, resulting in SNR ratios between 10 and 20 dB. For the noise generation, a second-order AR-process was used, generating a colored Gaussian noise. This simulates the physical EMG noise, which is a wide-band colored signal, whose dominant energy spans in the 50 – 150 Hz range.

Output SNR is calculated for the entire ECG signal as well as for the fragments delimiting the P wave, which is the most sensitive to noise (this last measure is denoted by PwSNR). For each input SNR the experience was repeated 10 times and the results were averaged. The output SNR is computed for each of ten wavelet mothers used in TIDWT, as well as for the signal resulted by averaging this ten versions of the denoised signal (the signal to the TIDWTED output). A selection of the results is shown in table 1.

For P wave region, an averaged PwSNR improvement factor was computed at each "overall" SNR (fig. 2). This factor represents the difference between the output and the input PwSNR. Note that in this case neither an input or output averaged PwSNR can be calculated, since for the same overall input SNR there is an important variation of the input PwSNR between different signals. The reason is that the ratio: energy of the P wave / energy of the whole beat is strongly dependent on the physical characteristics of the patient, so it's different for each patient in particular.

	Type	**INPUT SNR**					
		10	**12**	**14**	**16**	**18**	**20**
Output SNR	Coiflet 1	22.50	24.18	25.81	27.44	29.04	30.54
	Coiflet 2	22.49	24.21	25.84	27.52	29.16	30.64
	Coiflet 3	22.38	24.05	25.71	27.37	29.01	30.53
	Daubechies 4	22.42	24.05	25.72	27.34	28.98	30.49
	Daubechies 6	22.56	24.11	25.80	27.44	29.03	30.53
	Daubechies 8	22.45	24.10	25.74	27.40	29.02	30.54
	Daubechies 10	22.32	23.94	25.63	27.28	28.86	30.38
	Daubechies 12	22.23	23.84	25.43	27.07	28.72	30.27
	Symmlet 4	22.52	24.22	25.88	27.56	29.18	30.69
	Symmlet 6	22.41	24.08	25.73	27.41	28.71	30.57
	TIDWTED	22.58	24.27	25.96	27.59	29.18	30.70

Table 1: SNR improvement results.

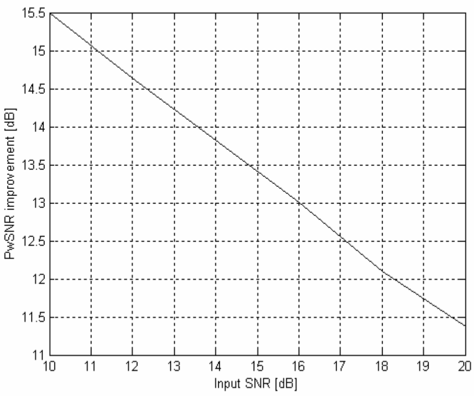

Fig. 2: PwSNR improvement factor versus overall input SNR.

The results shown in table 1 prove the effectiveness of the proposed method in terms of SNR improvement. This improvement is in all cases more than 10 dB. The performance is better compared to other ECG denoising results reported in literature [2,9,10], with spectacular differences for low SNRs. Yet, this comparison must be regarded with circumspection, since the work databases are different. The main gain in the SNR is brought by the use of a translation invariant wavelet transform (more than 1 dB better than DEDWT [6]). Note that in all cases, TIDWTED performs better than TIDWT with the best wavelet mother. Table 1 shows that there isn't a wavelet basis that can be classified as being "the best" for use in ECG denoising, since for different signals (and even for the same signal, but different SNRs) the best basis is different. In conclusion, even if the diversity enhancement obtained by the use of several different wavelet mothers does not significantly improve the results over TIDWT, it eliminates the performance's dependency on the choice of the wavelet mother.

The results in fig.2 illustrate excellent performance for the P wave denoising (more than 11dB PwSNR improvement in all cases). Note that for higher SNRs, PwSNR improvement is less spectacular than in low SNR conditions. This tendency (observable, but less important for the overall SNR) can be caused by the reduced energy of the P wave, comparing with the overall beat energy. Thus, the shrinkage of the wavelet coefficients, even if adapted to different portions of the signal, could affect the useful P coefficients.

3.2.2 Denoising influence on signal segmentation

This set of simulations evaluates the denoising influence on the signal segmentation. This approach takes into account the fact that signal denoising is only a pre-treatment step in ECG processing, being followed by segmentation, relevant parameters computation and patients classification. It is strongly desirable that segmentation results do not be dependent on the pre-treatment applied. When the signals are strongly affected by noise and their automatic segmentation is virtually impossible, a good denoising method should accurately estimate the useful ECG trace, allowing for a reasonable segmentation. The segmentation method used in this paper was presented and implemented by the authors in [11]. The method captures the dependencies that exist between the wavelet coefficients situated at different decomposition levels in the form of a probabilistic Markov tree with hidden states. The hidden state is represented by the coefficient's energy.

In this respect, we applied our method as a pre-treatment step for 30 relatively clean signals from CHU Brest database, that were next segmented using the procedure in [11] (only the first 20 beats were considered). The segmentation results for P wave were compared with the case where another denoising procedure [12] is applied (a SURE filtering [13], followed by a Wiener filtering with the protection of the QRS coefficients) (see table 2). The segmentation results, using the denoising proposed in [6] (the same algorithm, but comparing two different transforms-TIDWT with one wavelet mother and DEDWT) are also illustrated in the table.

	Onset error	End error	Segmentation Error Rate
Method in [12]	11.16 ms	11.37 ms	15.96 %
DEDWT	11.01 ms	8.87 ms	15.2 %
TIDWT	10.22 ms	7.99 ms	13.46 %
TIDWTED	10.64 ms	8.09 ms	13.54%

Table 2 : Denoising effects on the automatic segmentation of the P wave.

The results are quasi similar to those obtained by using TIDWT. The improvement is significant with respect to [12], showing a 2.5% reduction of the segmentation error rate. Note that a segmentation is considered erroneous if at least one of the three conditions are met: onset error>25ms, offset error>25 ms, more than 10 P wave with segmentation error for one single patient. The reference segmentation was provided by cardiologists from CHU Brest. Note that, unlike in the SNR improvement case, diversity enhancement in TIDWT does not improve the segmentation results. This could indicate that there are certain wavelet basis that are better for the segmentation than others.

In order to provide a deeper analysis of the denoising influence on the segmentation in various SNR conditions, another set of tests was performed. The test procedure has three steps: artificially generated noise is added on five signals with reduced segmentation error, the denoising procedure is applied and the segmentation is repeated, this time on the denoised signal. This way, a comparison of the segmentation results for the original and denoised signals can be done. In fig. 3, two extreme cases are shown. The worst case (test signal number 1) corresponds to a low-energy P wave, (input PwSNR= -6.69dB, for an overall input SNR of 10 dB). In this case, the denoising assures acceptable signal segmentation errors from input SNRs superior to 12 dB (input PwSNR<-4dB), which is a promising result. In the best case (prominent P wave), the denoising has little effect on the segmentation error, since the amount of noise is not sufficiently large to perturb the segmentation. In this situation, segmentation shift is reduced from the beginning.

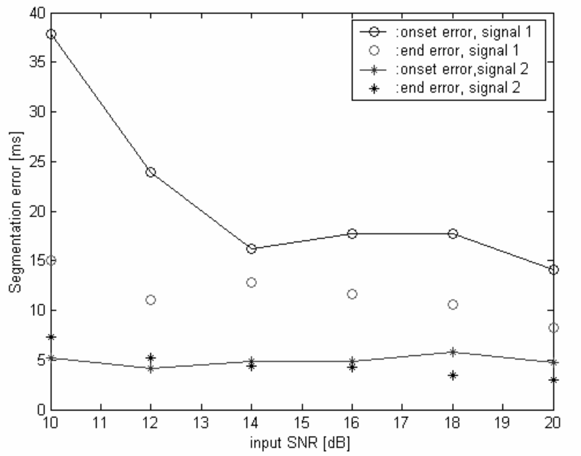

Fig. 3: Segmentation errors in various SNR conditions: two illustrative examples.

3.2.3 Denoising of signals affected by real noise

The algorithm was conceived for the denoising of ECG signals affected by real noise. This pre-treatment should allow correct signal segmentation. For testing our algorithm's effectiveness in real conditions, we applied it on a high number of ECG signals strongly perturbed by noise. The signals are raw data, provided by Task Force Monitor 3040i, from CNS Systems. An example is shown in figure 4.

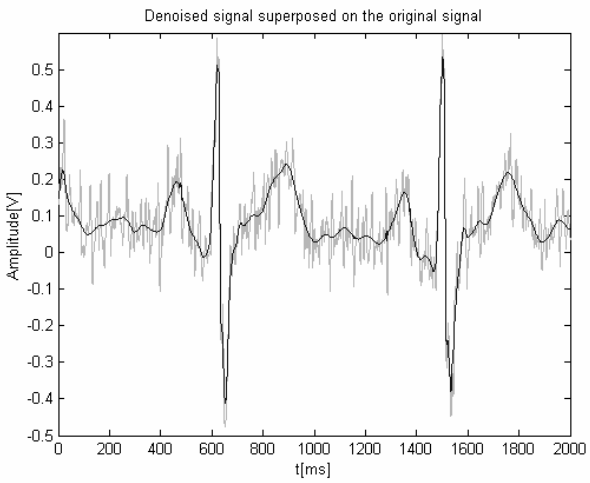

Fig. 4: Denoising applied on signal affected by real noise.

In the background, the original signal is strongly affected by noise. The denoised signal (in black) allows for a simple visual identification of the elementary waveforms (P, QRS,T). The noise is eliminated. A loss in amplitude of the QRS complex can be noticed, but this loss is maintained in a tolerable range (<10 % is acceptable, accordingly to cardiologists).

In conclusion, tests made on signals affected by real noise showed promising results. The noise that affects the signal in fig. 4 is a wide-band colored noise, which fits the theoretical background of the algorithm (see section II). On some test signals, the parasite component of 50 Hz can be clearly highlighted. Our algorithm has good practical results in these situations too.

IV. CONCLUSIONS

A new ECG denoising algorithm is presented in this paper. This algorithm relies on a modification of the empirical Wiener filtering in wavelet domain proposed in [3]. Superior performance is provided by a diversity enhancement of the signal to be processed. Realistic assumptions on the statistical properties of the useful wavelet coefficients are made. Several set of tests were performed, in order to demonstrate our algorithm's effectiveness. These tests highlight a good behavior of our method: an important SNR improvement (computed for signal affected by artificial noise), positive effect on signal segmentation and removal of noise (evaluated by a visual inspection of the denoised signal in real noise conditions).

Further improvements are still possible. In the future, we will focus on more elaborated methods for obtaining the pilot signal and in a deeper statistical study of the ECG wavelet coefficients.

ACKNOWLEDGEMENT

An important part of the research work that led to the redaction of this paper was supported and encouraged by Mister J. M. Boucher, professor at ENST Bretagne. Furthermore, ECG database, collected at CHU Brest, is obtained by his gentleness. That's why the authors would like to gratefully thank Mister Boucher for his help and support.

REFERENCES

[1] L. Donoho, **"**Denoising by Softthresholding", *IEEE Trans. Inform. Theory,* 41:613-627,1995.

[2] N. Nikolaev and A. Gotchev, "ECG signal denoising using wavelet domain Wiener filtering", *Proceedings of the European Signal Processing Conf. EUSIPCO-2000*, Tampere, Finland, pp. 51-54, September 2000.

[3] S. Ghael, A. Sayeed and R. Baraniuk, "Improved wavelet denoising via empirical Wiener filtering", *Proceedings of SPIE 3169*, San Diego, U.S.A., July 1997.

[4] A. and D. Isar, "Adaptive denoising of low SNR signals", *ARI, The Bulletin of the Istanbul Technical University*, Volume 53, Number 2, pp. 31-37, September 2003.

[5] R. Coifman and D. Donoho, "Transaltion-invariant de-noising", *Wavelets and Statistics*, A. Antoniadis and G. Oppenheim Eds, Springer-Verlag, pp. 125-150, New York, 1995.

[6] M. Oltean, J.M. Boucher and A. Isar, "MAP filtering in wavelet domain applied to ECG signal denoising", accepted to *ICASSP 2006, Toulouse*, France.

[7] L. Sendur and I. Selesnick, "Bivariate shrinkage functions for wavelet based denoising exploiting interscale dependency", *IEEE Trans. Signal Proc.*, vol. 50, pp. 2744-2756, Nov. 2002.

[8] I. Jonstone et B. Silverman, "Wavelet threshold estimators for data with correlated noise", *Journal of Royal Statistical Society*, Volume B 59, number 2, pp. 319-351, 1997.

[9] A. Gotchev, N. Nikolaev and K. Egiazarian, "Improving the transform domain ECG denoising performance by applying inter-beat and intra-beat decorrelating transforms", *Proceedings of the IEEE International Symposium on Circuits and Systems, ISCAS 2001*, Volume II, pp. 17-20, Sydney, Australia, May 2001.

[10] N. Nikolaev, A. Gotchev, "De-noising of ECG signals using wavelet shrinkage with time-frequency dependant threshold", *Proceedings of the European Signal Processing Conf. EUSIPCO-98*, Island of Rhodes, Greece, September 1998, pp. 2449-2453.

[11] S. Graja and J.M. Boucher, "Multiscale Hidden Markov Model Applied to ECG Segmentation", *Intelligent Signal Processing IEEE International Symposium* , pp 105-109, Sept. 2003.

[12] R. Le Page, *Detection and analysis of the P wave of an ECG signal: application to the atrial fibrillation detection,* Ph. D.Thesis, Université de Bretagne Occidentale, Brest, France, February 2003.

[13] D. Donoho and I. Johnstone, "Adapting to unknown smoothness via wavelet shrinkage", *Journal of American Statistical Assoc.*, vol. 90, pp. 1200-1224, December 1995.