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## Scattering parameters of symmetrical networks

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**Abstract** – Following a recent publication, teaching of the scattering parameters is further investigated in this paper. It is shown that complete calculations, sometimes considered too complicated and therefore avoided, can be alleviated and made transparent, by exploiting the symmetry of the electrical networks involved.

**Keywords:** scattering parameters, electrical engineering education, microwaves

## I. INTRODUCTION

In a recent paper [1], the author identifies and corrects two deficiencies in the way scattering parameters are taught in microwaves courses: an insufficient number of calculation examples in textbooks, and the lack of an adequate physical interpretation for the results. For illustration purposes, he presents three "carefully selected examples", of which the first one is calculated by hand, and, for the other two, circuit- and microwave-oriented software packages are used (PSPICE and Microwave Office respectively). It turns out that these examples are also ideal for illustrating the simplifications to hand calculations when the topological symmetry of certain circuits is exploited, and therefore, complete solutions for the  $S$  parameters are presented here.

Hand calculations are important for understanding the behavior of circuits, which can influence beneficially system design, the interpretation of results, or finding errors when simulation software is used. Real life circuits are generally described by complicated algebraic expressions; therefore every method for alleviating their derivation is valuable.

All the examples deal with reciprocal networks, therefore the scattering matrix is symmetric  $S_{ij} = S_{ji}$ ,  $i \neq j$ . However, for illustration purposes, calculations will be performed for all the scattering parameters, and reciprocity will be used for verification.

## II. EXAMPLES OF SCATTERING PARAMETERS CALCULATION

The first example of a symmetrical, resistive network is reproduced from [1] in Fig. 1.

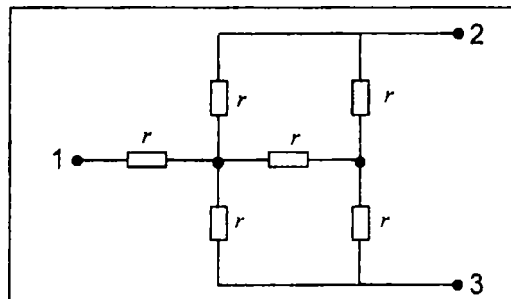


Fig. 1: A symmetric resistive network

All values for the resistances are normalized to the reference resistance (same value for all the 3 ports, even if for symmetry exploitation it is enough to have the same reference impedance only at ports 2 and 3). In order to calculate  $S_{11}$ ,  $S_{21}$ , and  $S_{31}$ , the network must be completed with the elements drawn with dashed lines in Fig. 2 (a). In [1] the scattering parameters are obtained using wye-to-delta transformations. However, a simpler way is to note that, due to symmetry, points A and B in Fig. 2 (a) are at the same potential. Consequently, they can be electrically connected without altering the voltages and currents at ports 1, 2, and 3. The equivalent network is represented in Fig. 2 (b), from which there results by inspection

$$r_{in,1} = r + \frac{r}{2} \parallel \left( r + \frac{r}{2} \right) + \frac{1}{2} = \frac{11r+4}{8}, \quad (1)$$

and

$$S_{11} = \frac{r_{in,1} - 1}{r_{in,1} + 1} = \frac{11r - 4}{11r + 12}. \quad (2)$$

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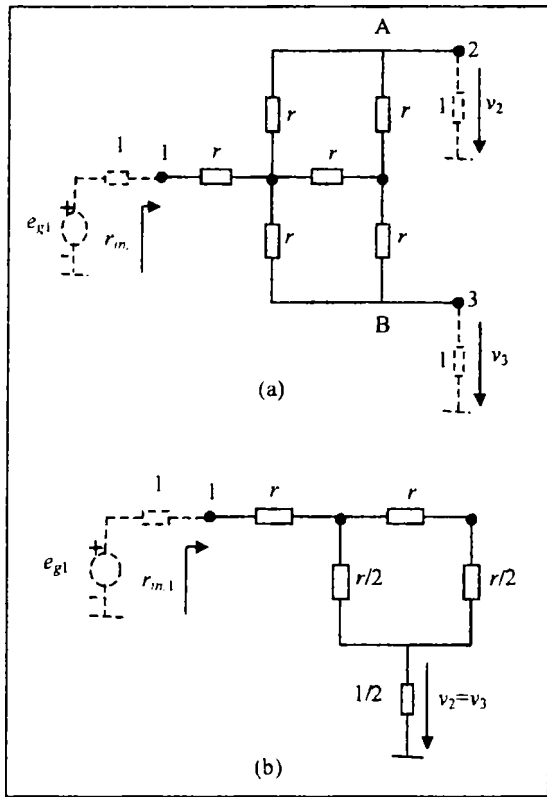


Fig. 2: a) The network in Fig. 1 with additional components for the computation of the reflection coefficient  $S_{11}$  at port 1; b) equivalent circuit due to symmetries.

Also due to symmetry, and considering the computation formula (9) in [1],

$$S_{21} = S_{31} = \frac{2v_2}{e_{g1}} = \frac{2 \times \frac{1}{2}}{1 + r + \frac{r}{2} \parallel \left( \frac{r}{2} + \frac{1}{2} \right)} = \frac{8}{11r + 12} \quad (3)$$

For the rest of the parameters, the network in Fig. 3 (a) can be used. In order to apply the definition, for calculating  $S_{22}$ ,  $S_{12}$ , and  $S_{32}$  it must be imposed  $e_{g3} = 0$ , while for  $S_{33}$ ,  $S_{13}$ , and  $S_{23}$  one must choose  $e_{g2} = 0$ . Of course, in this case  $S_{12} = S_{13}$  due to symmetry, and the fact that  $S_{23} = S_{32}$  results not only from symmetry, but also from reciprocity. Advantage will be taken of symmetry for avoiding again wye-to-delta transformations. Inspiration comes from the even and odd modes on coupled transmission lines [2], or common mode and differential mode in differential amplifiers [3]. The following method is also used in [4] and tackled in [5]. Two situations will

be considered: first, the "even" case,  $e_{g2} = e_{g3} = \frac{v}{2}$ , (for which "+" will be used), and second, "odd" case,  $e_{g2} = -e_{g3} = \frac{v}{2}$ .

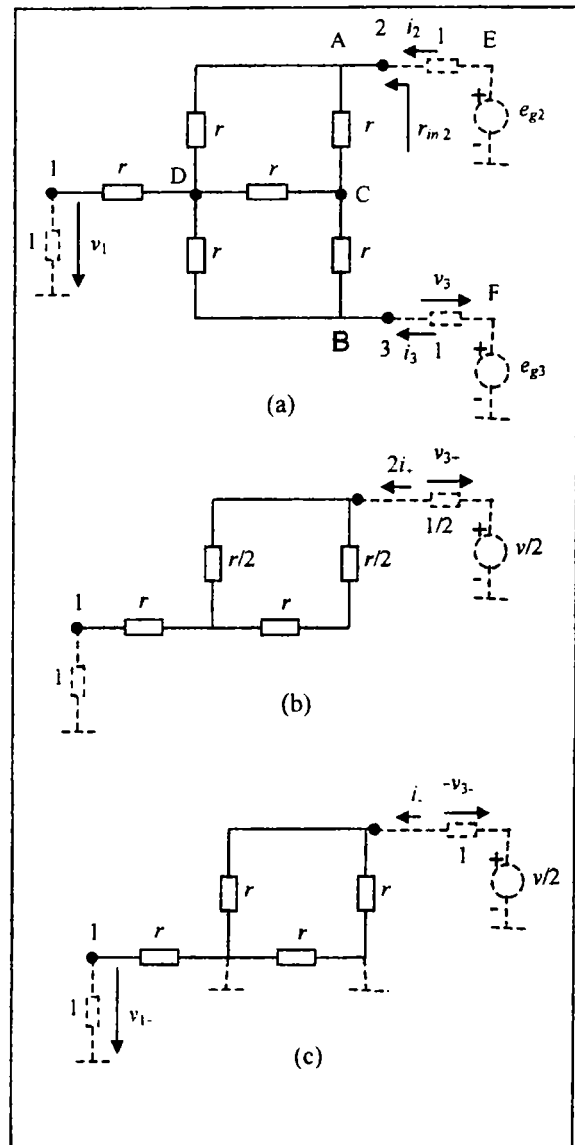


Fig. 3: a) The network in Fig. 1 with additional components for the computation of the  $S_{22}$ ,  $S_{12}$ ,  $S_{32}$  coefficients; b) equivalent circuit, "even" excitation; c) equivalent circuit, "odd" excitation.

case,  $e_{g2} = -e_{g3} = \frac{v}{2}$  (denoted by "-"). In the even case, points A and B in Fig. 3 (a) are at the same potential, and can be electrically connected, and similarly points E and F. The resulting network is represented in Fig. 3 (b). In the odd mode, points D and C in Fig. 3 (a) are at the ground potential, and the network schematic can be redrawn as in Fig. 3 (c). By inspection, it can be written

$$v_{1+} = \frac{v}{2} \frac{1}{\frac{1}{2} + \frac{r}{2} \parallel \left( \frac{r}{2} + r \right) + r + 1} = \frac{4v}{11r + 12}, \quad v_{1-} = 0,$$

$$v_{3-} = -\frac{v}{2} + \frac{v}{2} \frac{\frac{r}{2} \parallel \left( \frac{r}{2} + r \right) + r + 1}{\frac{1}{2} + \frac{r}{2} \parallel \left( \frac{r}{2} + r \right) + r + 1} = -\frac{v}{2} \frac{4}{11r + 12}.$$

$$v_{3+} = \frac{v}{2} \left( 1 - \frac{\frac{r}{2}}{1 + \frac{r}{2}} \right) = \frac{v}{2} \frac{r}{r + 2},$$

$$2i_- = \frac{v/2}{\frac{1}{2} + \frac{r}{2} \parallel \left( \frac{r}{2} + r \right) + r + 1} = \frac{4v}{11r + 12},$$

$$i_- = \frac{\frac{v}{2}}{1 + \frac{r}{2}} = \frac{v}{r + 2}.$$

Due to linearity, for  $e_{g2} = v$  and  $e_{g3} = 0$ , the values for the electrical quantities defined in Fig. 3 (a) can be obtained from  $v_1 = v_{1+} + v_{1-}$ ,  $v_3 = v_{3+} + v_{3-}$ ,

$i_2 = i_- + i_+$ , and  $r_{in,2} = \frac{v}{i_2} - 1$ . Finally, the following scattering parameters result:

$$S_{22} = \frac{r_{in,2} - 1}{r_{in,2} + 1} = \frac{11r^2 + 8r - 8}{11r^2 + 34r + 24}; \quad (4)$$

$$S_{12} = \frac{2v_1}{v} = \frac{8}{11r + 18}; \quad (5)$$

$$S_{32} = \frac{2v_3}{v} = \frac{18r + 16}{(11r + 12)(r + 2)}. \quad (6)$$

The second example is the network represented with solid line in Fig. 4(a). The circuit contains the normalized impedances  $z_1$  and  $z_2$ , which are respectively an inductance and a capacitance [1]. Due to symmetry, it is sufficient to find the  $S$  parameters at only one of the four ports. The elements it must be completed with in order to calculate  $S_{i1}$ ,  $i=1 \dots 4$  are drawn with dashed line in Fig. 4 (a). The same symmetry allows for the electrical connection of points A and B, and the equivalent network represented in Fig. 4 (b) is obtained, where  $z_2' = 1 \parallel z_2$ . There results at a glance

$$z_{in,1} = z_2 \parallel \left[ \frac{z_1}{2} + \frac{z_2'}{2} \parallel \left( \frac{z_1}{2} + z_2' \right) \right],$$

$$v_2 = e_{g1} \frac{z_{in}}{1 + z_{in}} \frac{\left( \frac{z_2'}{2} \right) \parallel \left( \frac{z_1}{2} + z_2' \right)}{\frac{z_1}{2} + \left( \frac{z_2'}{2} \right) \parallel \left( \frac{z_1}{2} + z_2' \right)},$$

$$v_3 = v_2 \frac{z_2'}{\frac{z_1}{2} + z_2'}.$$

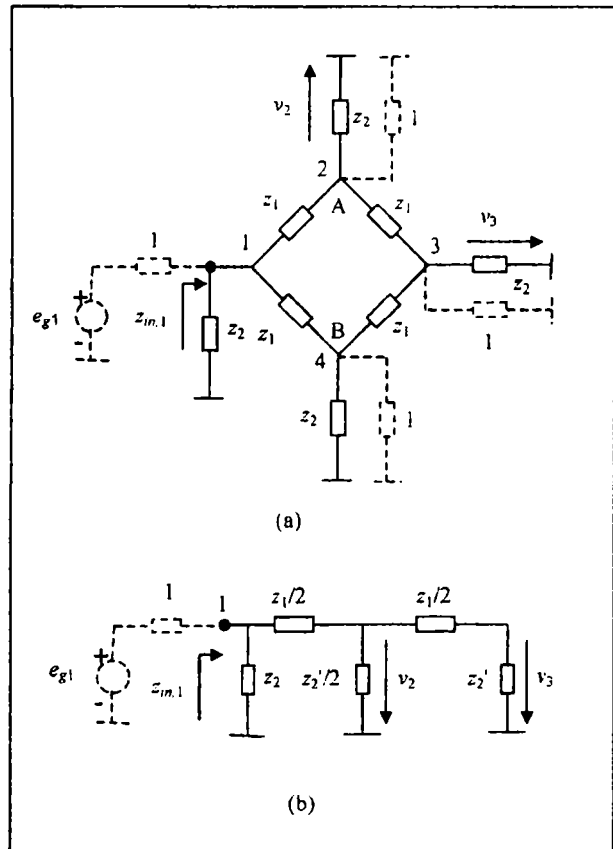


Fig. 4: a) Symmetric four-port reactive circuit with additional components for  $S$  parameters computation (dashed line), b) equivalent circuit.

Now, the formulas for the scattering parameters [1] can be applied:

$$S_{11} = \frac{z_{in,1} - 1}{z_{in,1} + 1} = \frac{(z_1 z_2 - z_1 - 2z_2)(z_1 + 3z_2 + z_1 z_2)}{(z_1 z_2 + z_1 + 2z_2)(z_1 + 4z_2 + z_1 z_2)} + \frac{z_2(z_2 - 1)}{(z_2 + 1)(z_1 + 4z_2 + z_1 z_2)} \quad (7)$$

$$S_{21} = \frac{2v_2}{e_{g1}} = \frac{2z_2^2}{(1 + z_2)(z_1 + 4z_2 + z_1 z_2)}, \quad (8)$$

$$S_{31} = \frac{2v_3}{e_{g1}} = S_{21} \frac{2z_2}{z_1 + 2z_2 + z_1 z_2}. \quad (9)$$

A general purpose, non circuit- or microwave-oriented software package, such as Matlab, can be used for plotting the insertion loss, return loss or isolation for various values of the elements in the circuit.

In figs. 9 and 10 of [1], the S parameters seem to be evaluated at a few frequency points and the response in the whole frequency range is obtained by an interpolation. Usually, this approach is used when determination of the response in one point requires long computation time.

The proposed method can be used both for the determination of the response of the circuit at few sampling points followed by an interpolation scheme, or to compute the response in the whole range with a relative large number of points.

The computation of the  $S_{11}$ ,  $S_{12}$ ,  $S_{13}$  was implemented in a Matlab script. Its run on a Pentium III with 1GHz clock required 0.01 s for 1000 sampling points.

As a third example, a symmetric Wilkinson power divider [1] is schematically represented with solid line in Fig. 5 (a): on the same figure, with dashed lines the circuits needed for the calculation of the S parameters, are also represented. All the impedances are normalized to a given reference impedance, here considered identical for the three ports. The transmission lines in the circuit are supposed identical, with (normalized) characteristic impedance  $z_c$ , propagation constant  $\gamma = \alpha + j\beta$  and length  $L$ .

For the calculation of  $S_{11}$ ,  $S_{21}$ , and  $S_{31}$ , one must take  $e_{g2} = e_{g3} = 0$ . Due to symmetry, the voltages over the lines are equal at equally spaced points from the generator; consequently, there is no current flow through the resistance  $r$ , and the two transmission lines can be conceptually connected in parallel, as in Fig. 5 (b). The result is a transmission line with the same propagation constant, but with a characteristic impedance  $z_c/2$ .

(Indeed, consider a standard equivalent circuit of a differential length of transmission line, [2, p. 86]. Conceptually connecting two such circuits in parallel, node by node, in the case of identical, corresponding voltages and currents, reduces by a factor of two the line resistance and the line inductance, and doubles the line capacitance and the line conductance. The substitution of these values in the expressions of the characteristic impedance and the propagation constant [2, p. 88] yields the above stated result.)

The normalized input impedance in Fig. 5 (b) is

$$z_{in,1} = \frac{z_c \left( \frac{1}{2} + \frac{z_c}{2} \tanh(\gamma L) \right)}{\frac{z_c}{2} + \frac{1}{2} \tanh(\gamma L)}, \quad (10)$$

and, consequently

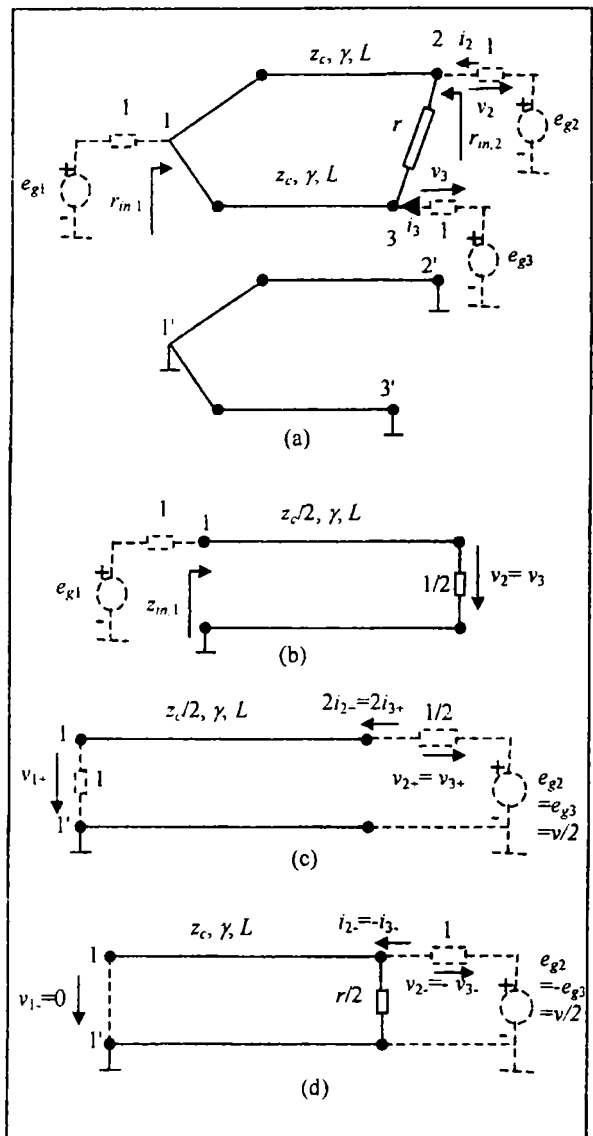


Fig. 5. (a) The Wilkinson power divider (solid line) and the components needed for the calculation of the scattering parameters (dashed line); (b) equivalent circuit for the calculation of the scattering parameters at port 1; (c) "even" excitation; (d) "odd" excitation.

$$S_{11} = \frac{z_{in,1} - 1}{z_{in,1} + 1} = \frac{(z_c^2 - 2) \sinh(\gamma L) - z_c \cosh(\gamma L)}{(z_c^2 + 2) \sinh(\gamma L) + 3z_c \cosh(\gamma L)}. \quad (11)$$

In the following, it will be needed the use of an expression for the voltage on a transmission line of normalized characteristic impedance  $z_c$ , propagation constant  $\gamma$ , and length  $L$ , terminated on a normalized load impedance  $z_L$ , and connected to a generator  $e_g$ , of normalized internal impedance  $z_g$ . Such an expression can be found in most textbooks on transmission lines; the variant adopted here is

$$v(x) = e_g \frac{z_c}{z_g + z_c} \frac{e^{-\gamma(L-x)} + \Gamma_L e^{-\gamma(L+x)}}{1 - \Gamma_L \Gamma_g e^{-2\gamma L}}, \quad (12)$$

where  $x$  is the distance to the load, and  $\Gamma_L = \frac{z_L - z_c}{z_L + z_c}$ ,

$\Gamma_g = \frac{z_g - z_c}{z_g + z_c}$  are the Fresnel reflection coefficient to

the load and to the generator respectively.

The application of (12) at the load end for the transmission line ( $x=0$ ) in Fig. 5 (b) gives

$$v_2 = v_3 = \frac{e_g \frac{z_c}{2} (1 + \Gamma_L) e^{-\gamma L}}{1 + \frac{z_c}{2} \Gamma_g e^{-2\gamma L}}. \quad (13)$$

In (13) the reflection coefficients are  $\Gamma_L = \frac{1 - \frac{z_c}{2}}{1 + \frac{z_c}{2}}$ ,

and  $\Gamma_g = \frac{1 - \frac{z_c}{2}}{1 + \frac{z_c}{2}}$ . There follows

$$S_{21} = S_{31} = \frac{2v_2}{e_g} = \frac{2z_c}{(2 + z_c^2) \sinh(\gamma L) + 3z_c \cosh(\gamma L)}. \quad (14)$$

The scattering parameters relative to ports 2 and 3 are respectively identical. The equivalent diagram for the even mode ( $e_{g1} = 0, e_{g2} = e_{g3} = \frac{v}{2}$  in Fig. 6 (a)) is represented in Fig. 6 (c), and the equivalent diagram for the odd mode ( $e_{g1} = 0, e_{g2} = -e_{g3} = \frac{v}{2}$ ) is represented in Fig. 6 (d). In the first case, the two transmission lines have been connected together point by point, and in the second case, it has been taken into account the fact that port 1 appears short-circuited (terminals 1 and 1' are at the ground potential), due to symmetry.

In the even case, the reflection coefficients at the load

and the generator are  $\Gamma_{L+} = \frac{1 - \frac{z_c}{2}}{1 + \frac{z_c}{2}}$  and

$\Gamma_{g+} = \frac{1 - \frac{z_c}{2}}{1 + \frac{z_c}{2}}$  respectively. The application of (12)

at the load end and generator end yields

$$v_{1+} = \frac{v}{2} \frac{\frac{z_c}{2} (1 + \Gamma_{L+}) e^{-\gamma L}}{1 + \frac{z_c}{2} \Gamma_{g+} e^{-2\gamma L}},$$

$$v_{2+} = v_{3+} = \frac{v}{2} \frac{\frac{z_c}{2} (1 + \Gamma_{L+}) e^{-\gamma L}}{1 + \frac{z_c}{2} \Gamma_{g+} e^{-2\gamma L}} \frac{v}{2}.$$

Finally, from Fig. 5 (c) there results directly  $i_{2-} = i_{3-} = -v_{2+} = -v_{3+}$ .

In the odd case, by taking into account Thévenin's theorem applied to the generator, the reflection

coefficients are  $\Gamma_{L-} = -1$ ,  $\Gamma_{g-} = \frac{1 \parallel \frac{r}{2} - 1}{1 \parallel \frac{r}{2} + 1}$ , and

consequently, from (12) and Fig. 6 (d)

$$v_{1-} = 0,$$

$$v_{2-} = -v_{3-} = \frac{v}{2} \frac{\frac{r}{2} \frac{z_c}{2} (1 - \Gamma_{L-}) e^{-\gamma L}}{1 + \frac{r}{2} \Gamma_{g-} e^{-2\gamma L}} \frac{v}{2},$$

$$i_{2-} = -i_{3-} = -v_{2-} = v_{3-}.$$

As shown above, in connection to the second example, the electrical diagram needed for the calculation of the scattering parameters relative to port 2 is the superposition of the situations represented in Fig. 5 (c) and (d). For

example  $S_{12} = \frac{2v_{1-}}{v} = \frac{2(v_{1-} + v_{1+})}{v}$ ; performing the calculations leads to the right member of (14), as  $S_{12} = S_{21}$  due to reciprocity. Symmetry and reciprocity also imply  $S_{13} = S_{12} = S_{31}$ . New expressions will be obtained for  $S_{32} = S_{23}$ , and  $S_{22} = S_{33}$ .

In the first case

$$S_{32} = \frac{2v_{3-}}{v} = \frac{2(v_{3-} + v_{3+})}{v} =$$

$$= z_c \frac{2 \cosh(\gamma L) + z_c \sinh(\gamma L)}{(2 + z_c^2) \sinh(\gamma L) + 3z_c \cosh(\gamma L)} -$$

$$- \frac{r z_c}{r + (r + 2)z_c} \frac{(r + 1)(1 - e^{-2\gamma L})}{r + 1 - e^{-2\gamma L}}. \quad (15)$$

For  $S_{22}$  the input impedance at port 2 is needed:

$z_{in,2} = \frac{v_{2+} + v_{2-} + v}{i_{2+} + i_{2-}}$ . There results

$$\begin{aligned}
 S_{22} &= \frac{z_{in,2} - 1}{z_{in,2} + 1} = 1 + \frac{2(v_{2+} + v_{2-})}{v} = \\
 &= z_c \frac{2 \cosh(\gamma L) + z_c \sinh(\gamma L)}{(2 + z_c^2) \sinh(\gamma L) + 3z_c \cosh(\gamma L)} + \quad (16) \\
 &+ \frac{rz_c}{r + (r + 2)z_c} \frac{(r + 1)(1 - e^{-2\gamma L})}{r + 1 - e^{-2\gamma L}} - 1.
 \end{aligned}$$

A Matlab script was implemented again for the computation of the  $S_{11}$ ,  $S_{22}$ , and  $S_{23}$  parameters. Its run on a the same Pentium III with 1GHz clock required less than 0.1 s for 5000 sampling points.

### III. CONCLUSION

In this work, some educational applications of electrical networks symmetry for scattering parameters calculation was illustrated, using both lumped elements networks and networks containing transmission lines. Reflection and transmission coefficients have been calculated in closed form, in order to alleviate the interpretation of the results, and highlight the effect of various parameters.

Comparisons between results obtained by the proposed approach and those in [1], where they have been computed numerically, with dedicated software, are in good agreement,.

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