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CONNECTIONS BETWEEN SOME CONCEPTS OF POLYNOMIAL TRICHOTOMY FOR DISCRETE SKEW-EVOLUTION SEMIFLOWS IN BANACH SPACES

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Abstract

The present paper studies the property of trichotomy described by a polynomial behaviour according to decay, expansion and growth of the solution on the stable, unstable and central manifold respectively. [1]

Keywords and phrases: *skew-evolution semiflow; polynomial trichotomy, strong polynomial trichotomy and weak polynomial trichotomy.*

1 Introduction

The issue of decomposing the state space into a direct sum of subspaces, where the trajectories of the system define a prescribed behavior is triggered by the asymptotic behavior of first-order differential equations. The term of exponential trichotomy shapes the fact that the state space into three closed subspaces: stable subspace, unstable subspace and the so-called central manifold. While the stable subspace leads the pattern of the solution to converge (in norm) towards zero, and the unstable one to converge (in norm) towards infinity, on the central manifold the solutions need only to have polynomially growth and decay.

The trichotomy property is a natural generalization of the well-known dichotomy property of dynamical systems, refined as several results were published from which we point out the following: [1], [4], [5], [6], [7], [8] and [10]. The trichotomy property was first mentioned by Sacker and Sell in [9] and several results, related to polynomial trichotomy, were published in [1], [4], [5], [8], [10].

The present paper studies the property of trichotomy described by a polynomial behavior according to decay, expansion and growth of the solution on the stable, unstable and central manifold respectively. The links between the

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concepts presented in this paper (polynomial trichotomy, strong polynomial trichotomy and weak polynomial trichotomy) are indicated mostly with the aid of examples and counterexamples, which provide a set of systems having such properties on one hand, and clearly delimiting the concepts on the other hand.

2 Discrete evolution semiflows

Let (X, d) be a metric space, V a Banach space, and $\mathcal{B}(V)$ the Banach space of all bounded linear operators acting on V. We denote by $D = \left\{(m, n) \in \mathbb{N}^2 : m \geq n\right\}$.

Definition 2.1. A mapping $\varphi : D \times X \to X$ is called a *discrete evolution semiflow* on X if the following conditions hold:

(es1) $\varphi(m, m, x) = x,$ for all $(m, x) \in \mathbb{N} \times X$;

(es2) $\varphi(m, n, \varphi(n, p, x)) = \varphi(m, p, x),$ for all $(m, n), (n, p) \in D, x \in X$.

Definition 2.2. A mapping $\Phi : D \times X \to \mathcal{B}(V)$ is called a *discrete evolution cocycle* over the evolution semiflow φ if:

(ec1) $\Phi(m, m, x) = I,$ for all $m \geq 0, x \in X$.

(ec2) $\Phi(m, n, \varphi(n, p, x))\Phi(n, p, x) = \Phi(m, p, x),$ for all $(m, n), (n, p) \in D$ and for all $x \in X$.

If Φ is a discrete evolution cocycle over the discrete evolution semiflow φ, then the pair $C = (\varphi, \Phi)$, defined by $C : D \times X \times V \to X \times V$, $C(m, n, x, v) = (\varphi(m, n, x), \Phi(m, n, x)v)$ is called a discrete *skew-evolution semiflow* on $X \times V$.

Definition 2.3. An operator valued sequence $P : \mathbb{N} \to \mathcal{B}(V)$ is called a **sequence of projections** if $P_n P_n = P_n$ for all $n \in \mathbb{N}$, where $P_n = P(n)$.

3 Trichotomy cvadruples

We will denote by $V = l^2(\mathbb{N}, \mathbb{R})$ the Banach space containing all the real-valued sequences $v = (v_k)_{k \geq 0}$ having the property

$$\sum_{n=0}^{\infty} |v_n|^2 < \infty,$$

endowed with the norm $\|v\|_2 = \left(\sum\limits_{n=0}^{\infty} |v_n|^2\right)^{1/2}$.

Definition 3.1. A sequence of projections $P : \mathbb{N} \to \mathcal{B}(V)$ is called **polynomially bounded** if there exist $M \geq 1$ and $\gamma \geq 0$ such that $\|P_n\| \leq M(n+1)^\gamma$ for all $n \in \mathbb{N}X$. If $\gamma = 0$, we say that it is **bounded**.

Definition 3.2. Three sequences of projections $P, Q, R : \mathbb{N} \to \mathcal{B}(V)$ are called **supplementary** if for all $n \in \mathbb{N}$ we have $P_n + Q_n + R_n = I$.

In what follows, we will present two examples which will serve our main aim.

Example 3.1. Consider $V = l^2(\mathbb{N}, \mathbb{R})$ and $p : \mathbb{N} \to \mathbb{R}$ a non-decreasing sequence. For each $n \in \mathbb{N}$ we define $P_{1,n} : l^2(\mathbb{N}, \mathbb{R}) \to l^2(\mathbb{N}, \mathbb{R})$ by $P_{1,n}v = (y_k(n))_{k \geq 0}$, where $y_{3k}(n) = v_{3k} + p(n) \cdot v_{3k+1}$, $y_{3k+1}(n) = y_{3k+2}(n) = 0$, $k \in \mathbb{N}$.

We have that

$$P_{1,n} \in \mathcal{B}(l^2(\mathbb{N}, \mathbb{R}))$$

and for all $n \in \mathbb{N}$ we have that

$$\max\{1, p(n)\} \leq \|P_{1,n}\| \leq 1 + p(n).$$

Furthermore, we define the sequence of projections

$$Q_1 : \mathbb{N} \to \mathcal{B}(V)$$

by

$$Q_{1,n}v = (z_k(n))_{k \geq 0},$$

where

$$z_{3k}(n) = -p(n)v_{3k+1}, \ z_{3k+1}(n) = v_{3k+1}, \ z_{3k+2}(n) = 0, \ k \in \mathbb{N}.$$

The following hold:

$$|Q_{1,n}v\|_2 \leq \|Q_{1,m}v\|_2$$

$$\|Q_{1,n}v\|_2 = \sqrt{(1 + p(n)^2) \cdot \sum_{k=0}^{\infty} |v_{3k+1}|^2}.$$

Finally, we define

$$R_1 : \mathbb{N} \to \mathcal{B}(V)$$

by

$$R_{1,n}v = (w_k(n))_{k \geq 0},$$

where

$$w_{3k}(n) = w_{3k+1}(n) = 0, \quad w_{3k+2}(n) = v_{3k+2}, \ k \in \mathbb{N}.$$

We have that R_1 is bounded, with $\|R_{1,n}\| = 1$, for all $n \in \mathbb{N}$, $x \in X$ and in addition, the sequences P_1, Q_1 and R_1 are supplementary.

Example 3.2. Let $V = l^2(\mathbb{N}, \mathbb{R})$ and consider

$$P_2, Q_2, R_2 : \mathbb{N} \to \mathcal{B}(l^2(\mathbb{N}, \mathbb{R}))$$

defined by $P_{2,n}v = (y_k(n))_{k\geq 0}$, $Q_{2,n}v = (z_k(n))_{k\geq 0}$ and $R_{2,n}v = (w_k(n))_{k\geq 0}$, where, for $k \in \mathbb{N}$, $y_{4n}(n) = v_{4k}$, $y_{4k+1}(n) = y_{4k+2}(n) = y_{4k+3}(n) = 0$, $z_{4k}(n) = z_{4k+3}(n) = 0$, $z_{4k+1}(n) = v_{4k+1}$, $z_{4k+2}(n) = v_{4k+2}$, $w_{4k}(n) = w_{4k+1}(n) = w_{4k+2}(n) = 0$, $w_{4k+3}(n) = v_{4k+3}$. We have that P_2, Q_2 and R_2 are three supplementary sequences of projections, with $\|P_{2,n}\| = \|Q_{2,n}\| = \|R_{2,n}\| = 1$ for all $n \in \mathbb{N}$.

Given three supplementary sequences of projections P, Q, R and $C = (\Phi, \varphi)$ a discrete skew-evolution semiflow, we will say that (C, P, Q, R) is a **trichotomic cvadruple**.

Two examples of trichotomic cvadruples are given below.

Example 3.3. On $V = l^2(\mathbb{N}, \mathbb{R})$ consider the sequences of projections P_1, Q_1 and R_1 from Example 3.1. Let

$$\lambda : \mathbb{N} \to (0, \infty)$$

and

$$\Phi_1 : D \to \mathcal{B}(l^2(\mathbb{N}, \mathbb{R}))$$

given by

$$\Phi_1(m, n, x) = \frac{\lambda(n)}{\lambda(m)} \cdot P_{1,n} + \frac{\lambda(m)}{\lambda(n)} \cdot Q_{1,m} + R_{1,n}$$

for all $(m, n, x) \in D \times X$. Taking into account that, for all $m, n \in \mathbb{N}$ the following hold:

$$P_{1,m}P_{1,n} = P_{1,n} \quad \text{and} \quad Q_{1,m}Q_{1,n} = Q_{1,m}$$

it is easy to check that Φ_1 is a discrete skew-evolution co-cycle. Moreover we have that for all $(m, n, x) \in D \times X$,

$$\Phi_1(m, n, x)P_{1,n} = \frac{\lambda(n)}{\lambda(m)}P_{1,n},$$

$$\Phi_1(m, n, x)Q_{1,n} = \frac{\lambda(m)}{\lambda(n)}Q_{1,m},$$

$$\Phi_1(m, n, x)R_{1,n} = R_{1,n}.$$

Example 3.4. On $V = l^2(\mathbb{N}, \mathbb{R})$ let P_2, Q_2 and R_2 be the sequences of projections defined in Example 3.2. For $\psi : \mathbb{N} \to (0, \infty)$ we define

$$\Phi_2 : D \to \mathcal{B}(l^2(\mathbb{N}, \mathbb{R}))$$

by

$$\Phi_2(m, n, x)v = \begin{cases} (y_k(m, n))_{k \geq 0} & \text{if } m > n \\ v, & \text{if } m = n \end{cases}$$

where for all $k \in \mathbb{N}$ and $(m, n, x, v) \in D \times X \times l^2(\mathbb{N}, \mathbb{R})$,

$$y_{4k}(m, n) = \frac{\psi(n)}{\psi(m)} v_{4k},$$

$$y_{4k+1}(m, n) = \frac{\psi(m)}{\psi(n)} v_{4k+1},$$

$$y_{4k+2}(m, n) = 0$$

and

$$y_{4k+3}(m, n) = v_{4k+3}.$$

One can easily observe that (Φ_2, P_2, Q_2, R_2) a trichotomic cvadruple and for

$$(m, n, x) \in D \times l^2(\mathbb{N}, \mathbb{R})$$

we have that

$$\Phi_2(m, n, x)P_{2,n}v = (p_k(m, n))_{k \geq 0},$$

where

$$p_{4k}(m, n) = \frac{\psi(n)}{\psi(m)} v_{4k},$$

$$p_{4k+1}(m, n) = p_{4k+2}(m, n) = p_{4k+3}(m, n) = 0$$

and

$$\Phi_2(m, n, x)Q_{2,n}v = \begin{cases} (q_k(m, n))_{k \geq 0}, & m > n \\ (\rho_k(m, n))_{k \geq 0}, & m = n \end{cases}$$

is given by

$$q_{4k}(m,n) = q_{4k+2}(m,n) = q_{4k+3}(m,n) = 0,$$

$$q_{4k+1}(m,n) = \frac{\psi(m)}{\psi(n)} v_{4k+1}$$

and

$$\rho_{4k}(m,n) = \rho_{4k+3}(m,n) = 0,$$

$$\rho_{4k+1}(m,n) = v_{4k+1},$$

$$\rho_{4k+2}(m,n) = v_{4k+2},$$

for all $n \in \mathbb{N}$, and

$$\Phi_2(m,n,x)R_2(n)v = (r_k(m,n))_{k\geq 0},$$

where

$$r_{4k}(m,n) = r_{4k+1}(m,n) = r_{4k+2}(m,n) = 0,$$

$$r_{4k+3}(m,n) = v_{4k+3}.$$

In what follows, we will present the main concepts of trichotomy, which will be studied and delimited in the remaining sections.

4 Concepts of discrete polynomial trichotomy

Definition 4.1. A trichotomic cvadruple (C, P, Q, R) is called **polynomiallty trichotomic** (p.t) if there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that for all $(m,n,x) \in D \times X$,

(pt$_1$) $(m+1)^\alpha \|\Phi(m,n,x)P_n\| \leq N(n+1)^{\alpha+\beta}$;

(pt$_2$) $(m+1)^\alpha \leq N(m+1)^\beta (n+1)^\alpha \|\Phi(m,n,x)Q_n\|$;

(pt$_3$) $(n+1)^\alpha \|\Phi(m,n,x)R_n\| \leq N(m+1)^\alpha (n+1)^\beta$;

(pt$_4$) $(n+1)^\alpha \leq N(m+1)^{\alpha+\beta}\|\Phi(m,n,x)R_n\|$.

If $\beta = 0$, then we say that (C, P, Q, R) is **uniformly polynomially trichotomic** (u.p.t).

Remark 4.1. If (C, P, Q, R) is (p.t) with constants N, α, β then

$$\max\{\|P_n\|, \|Q_n\|, \|R_n\|\} \leq 3N(n+1)^\beta, \quad \forall n \in \mathbb{N}.$$

Remark 4.2. If (C, P, Q, R) is (u.p.t) then it is also (p.t). The converse is not generally true. Consider, for example, the trichotomic cvadruple (Φ_1, P_1, Q_1, R_1) from Example 3.3 with $p(n) = \lambda(n) = n+1$. It is easy to check that (Φ_1, P_1, Q_1, R_1) is (p.t), but it cannot be (u.p.t), because P is not bounded.

Definition 4.2. A trichotomic cvadruple (C, P, Q, R) is said to be **strongly polynomially trichotomic** (s.p.t) if there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

(spt$_1$) $(m+1)^\alpha\|\Phi(m,n,x)P_n v\| \leq N(n+1)^{\alpha+\beta}\|P_n v\|$;

(spt$_2$) $(m+1)^\alpha\|Q_n v\| \leq N(m+1)^\beta(n+1)^\alpha\|\Phi(m,n,x)Q_n v\|$;

(spt$_3$) $(n+1)^\alpha\|\Phi(m,n,x)R_n v\| \leq N(m+1)^\alpha(n+1)^\beta\|R_n v\|$;

(spt$_4$) $(n+1)^\alpha\|R_n v\| \leq N(m+1)^{\alpha+\beta}\|\Phi(m,n,x)R_n v\|$

for all $(m, n, x, v) \in D \times X \times V$.

If $\beta = 0$, then we say that (C, P, Q, R) is **uniformly strongly polynomially trichotomic** (u.s.p.t).

Remark 4.3. If (C, P, Q, R) is (u.s.p.t) then it is also (s.p.t). The converse is not generally true, fact shown by Example 5.1.

Remark 4.4. If (C, P, Q, R) is (s.p.t) then for all $(m, n, x) \in D \times X$ one has that $Range\ Q_n \cap Ker\ \Phi(m,n,x) = Range\ R_n \cap Ker\ \Phi(m,n,x) = \{0\}$.

Definition 4.3. A trichotomic cvadruple (C, P, Q, R) is said to be **weakly polynomially trichotomic** (w.p.t) if there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

(wpt$_1$) $(m+1)^\alpha\|\Phi(m,n,x)P_n\| \leq N(n+1)^{\alpha+\beta}\|P_n\|$;

(wpt$_2$) $(m+1)^\alpha\|Q_n\| \leq N(m+1)^\beta(n+1)^\alpha\|\Phi(m,n,x)Q_n\|$;

(wpt$_3$) $(n+1)^\alpha\|\Phi(m,n,x)R_n\| \leq N(m+1)^\alpha(n+1)^\beta\|R_n\|$;

(wpt$_4$) $(n+1)^{\alpha}\|R_n\| \leq N(m+1)^{\alpha+\beta}\|\Phi(m,n,x)R_n\|$

for all $(m,n,x) \in D \times X$.

If $\beta = 0$ then we say that (C,P,Q,R) is **uniformly weakly polynomially trichotomic** (u.w.p.t).

Remark 4.5. If (C,P,Q,R) is (u.w.p.t) then it is also (w.p.t). The converse is not generally true, fact illustrated by Example 5.2.

In what follows, the connections between the above defined concepts are presented.

Remark 4.6. If a trichotomic cvadruple (C,P,Q,R) is (s.p.t) then it is also (w.p.t). Moreover, if (C,P,Q,R) is (u.s.p.t), then it is also (u.w.p.t).

Proposition 4.1. *Let (C,P,Q,R) be a trichotomic cvadruple. If (C,P,Q,R) is (p.t) then it is also (w.p.t). Moreover, (u.p.t) $\Rightarrow$ (u.w.p.t).*

Proof. It follows the reasoning from Proposition 3.11 from [2]. $\square$

Remark 4.7. Example 5.3 shows that (s.p.t) does not imply (p.t) and (u.s.p.t) does not imply (u.p.t). Example 5.4 shows that the concepts of (p.t) and (w.p.t) do not coincide. Example 5.5 shows that (p.t) doesn't imply (s.p.t) and (u.p.t) doesn't imply (u.s.p.t). Finally, Example 5.6 shows that (w.p.t) doesn't imply (s.p.t) and (u.w.p.t) doesn't imply (u.s.p.t).

Remark 4.8. The connections between the above enumerated concepts, taking into account the presented results, and the examples from the next section, are illustrated by the following diagram:

$$\begin{array}{ccccccc}
\boxed{\text{u.p.t}} & \overset{\nLeftarrow}{\nRightarrow} & \boxed{\text{u.s.p.t}} & \overset{\nLeftarrow}{\Rightarrow} & \boxed{\text{u.w.p.t}} & \overset{\Leftarrow}{\nRightarrow} & \boxed{\text{u.p.t}} \\
\not\Uparrow\Downarrow & & \not\Uparrow\Downarrow & & \not\Uparrow\Downarrow & & \not\Uparrow\Downarrow \\
\boxed{\text{p.t}} & \overset{\nLeftarrow}{\nRightarrow} & \boxed{\text{s.p.t}} & \overset{\nLeftarrow}{\Rightarrow} & \boxed{\text{w.p.t}} & \overset{\Leftarrow}{\nRightarrow} & \boxed{\text{p.t}}
\end{array}$$

5 Examples and counterexamples

Example 5.1. We will consider a simplified example. On $V = \mathbb{R}^3$, endowed with the canonical norm, consider $P,Q,R : \mathbb{N} \to \mathcal{B}(V)$ the sequences of constant canonical projections on $\mathbb{R}^3$, on the first, second and third coordinate respectively. We define, for all $(m,n,x) \in D \times X$:

$$\Phi(m,n,x) = \frac{(n+1)^{1+a_n}}{(m+1)^{1+a_m}}P_n + \frac{m+1}{n+1}Q_n + R_n,$$

where $a_n = \chi_{2\mathbb{N}+1}(n)$, $n \in \mathbb{N}$ (χ_A denotes the characteristic function of the set A). It is easy to see that (Φ, P, Q, R) is a trichotomic cvadruple which is (s.p.t) with $N = \alpha = \beta = 1$. But, if we would assume that (Φ, P, Q, R) is (u.s.p.t), then, in particular, there exist $N \geq 1$ and $\alpha > 0$ such that for all $(m, n, x, v) \in D \times X \times V$, we have that

$$(m+1)^{\alpha} \|\Phi(m, n, x) P_n v\| \leq N(n+1)^{\alpha} \|P_n v\|.$$

Let

$$v = (1, 0, 0) \in Range P_n$$

and $k \in \mathbb{N}$. Fix $x \in X$ and choose $m = 2k + 2$ and $n = 2k + 1$. The above inequality yields the following contradiction:

$$2k + 2 \leq N \left(\frac{2k+2}{2k+3} \right)^{\alpha - 1}$$

for all $k \in \mathbb{N}$.

Example 5.2. Let (Φ_1, P_1, Q_1, R_1) be as in Example 5.1. According to Remark 4.6 we have that (Φ_1, P_1, Q_1, R_1) is (w.p.t). The same contradiction is obtained, as in Example 5.1, by assuming that (Φ_1, P_1, Q_1, R_1) is (u.w.p.t).

Example 5.3. Let (Φ_1, P_1, Q_1, R_1) the trichotomic cvadruple from Example 3.3 cu $p(n) = (n+1)^{n+1}$ and $\lambda(n) = n + 1$. From the following estimations

$$(m+1)\|\Phi_1(m, n, x) P_{1,n} v\|_2 = (n+1)\|P_{1,n} v\|_2$$

$$(m+1)\|Q_{1,n} v\|_2 \leq \lambda(m)\|Q_{1,m} v\|_2 = (n+1)\|\Phi_1(m, n, x) Q_{1,n} v\|_2$$

$$(n+1)\|\Phi(m, n, x) R_n v\|_2 \leq (n+1)(m+1)\|R_n v\|_2$$

$$(n+1)\|R_n v\|_2 \leq N(m+1)^2 \|\Phi(m, n, x) R_n v\|_2$$

valid for all $(m, n, x, v) \in D \times X \times V$, we can see that (Φ_1, P_1, Q_1, R_1) is (u.s.p.t), hence it is also (s.p.t).
Assume by a contradiction that (Φ_1, P_1, Q_1, R_1) is (p.t). Then, according to Remark 4.1, we have that there exist $M \geq 1$, $\gamma \geq 0$ such that $\|P_{1,n}\| \leq M(n+1)^{\gamma}$, or all $n \in \mathbb{N}$. This leads us to $(n+1)^{n+1} = p(n) \leq \|P_{1,n}\| \leq M(n+1)^{\gamma}$. We conclude that (Φ_1, P_1, Q_1, R_1) is not (p.t) hence not (u.p.t) as well.

Example 5.4. Let (Φ_1, P_1, Q_1, R_1) the trichotomic cvadruple from Example 5.3. According to Remark 4.6, we have that (Φ_1, P_1, Q_1, R_1) is (u.w.p.t), hence it is also (w.p.t). But, by Example 5.3, it is not (p.t), nor (u.p.t).

Example 5.5. Let (C, P_2, Q_2, R_2) the trichotomic cvadruple from Example 3.4 with $\psi(n) = n + 1$. This leads us easily to the fact that that (C, P_2, Q_2, R_2) is (u.p.t).

In what follows, we will show that (C, P_2, Q_2, R_2) is not (s.p.t), and from here, it cannot be neither (u.s.p.t). Assume, by a contradiction, that (C, P_2, Q_2, R_2) is (s.p.t). Let $v = (v_k)_{k\geq 0}$ given by $v_{4k+2} = \frac{1}{4k+2}$, $v_{4k+3} = v_{4k+1} = v_{4k} = 0$, $k \in \mathbb{N}$. Obviously $v \in l^2(\mathbb{N}, \mathbb{R})$ and by denoting, for every $n \in \mathbb{N}$, $Q_{2,n}v = (z_k(n))_{k\geq 0}$, where $z_{4k}(n) = z_{4k+1}(n) = x_{4k+1} = z_{4k+3} = 0,\ z_{4k+2}(n) = v_{4k+2} = \frac{1}{4k+2}$, we can easily see that $(z_k(n))_{k\geq 0}$ is a nonzero sequence. Let now $(m, n, x) \in D \times X$ be with $m > n$. By denoting

$$\Phi_2(m, n, x)Q_{2,n}v = (q_k(m, n))_{k\geq 0},$$

with

$$q_{4k}(m, n) = q_{4k+1}(m, n) = \frac{m+1}{n+1}v_{4k+1} = q_{4k+2}(m, n) = q_{4k+3}(m, n) = 0,$$

it follows that $\Phi_2(m, n, x)Q_{2,n}v = 0$, which contradicts the facts proven in Remark 4.4, hence (C, P_2, Q_2, R_2) is not (s.p.t).

Example 5.6. Let (C, P_2, Q_2, R_2) the trichotomic cvadruple from Example 5.5. Taking into account that for all $n \geq 0$,

$$\|P_{2,n}\| = \|Q_{2,n}\| = \|R_{2,n}\| = 1,$$

it follows that (C, P_2, Q_2, R_2) is (u.w.p.t), hence (w.p.t). Again, by Example 5.5, we obtain that (C, P_2, Q_2, R_2) is not (s.p.t), hence it is neither (u.s.p.t).

References

[1] M.-G. Babuţia, T. Ceauşu, N. M. Seimeanu, *On Uniform Exponential Dichotomy of Evolution operators*, An. Univ. Timişoara Ser. Math. Inform. L, 2 (2012), 3-14.

[2] M.-G. Babuţia, N. M. Seimeanu, T. Ceauşu, *Polinomyal dichotomies for evolution operators through examples and counterexamples*, Bull. Instit. Politehn. Iaşi, Tom LX (LXIV), Fasc, 4, 2014, s31-52.

[3] D. Borlea, *Polynomial trichotomy for skew-evolution semiflows on Banach spaces*, Proceedings of the International Symposium - Research and Education in Innova Era, Section Mathematics and Computer Science, 6th Edition, Aurel Vlaicu University of Arad Publishing House (2016), 38-47.

[4] L. Barreira, C. Valls, *Polynomial growth rates*, Nonlinear Anal.-Theor. 71 (2009), 5208-5219.

[5] L. Barreira, M. Fan, C. Valls, J. Zhang, *Robustness of nonuniform polynomial dichotomies for difference equations*, Topol. Methods Nonlinear Anal. 37 (2011), no. 2, 357-376.

[6] M. I. Kovacs, M. G. Babutia, M. Megan, *On (h,k)-dichotomy in Banach spaces* , Bull. Ştiinţ Univ. Politehn. Timiş. Ser. Mat.-Fiz. 59 (73) (2014), Fasc. 1/2014, 19-27.

[7] M. Megan, A. L. Sasu, B. Sasu, *Discrete admissibility and exponential dichotomy for evolution families*, Discrete Contin. Dynam. Systems 9 (2003), 383-397.

[8] M. L. Rămneanţu, T. Ceauşu, M. Megan, *On nonuniform polynomial dichotomy of evolution operators in Banach spaces*, Int. J. Pure Appl. Math. 75 (2012), 305-318.

[9] R. J. Sacker, G. R. Sell, *Existence of dichotomies and invariant splittings for linear differential systems*, J. Diff. Equat. 15(3) (1974), 429-258.

[10] A. L. Sasu, B. Sasu, *Discrete admissibility and exponential trichotomy of dynamical systems*, Discrete Contin. Dyn. Syst. 34 (7) (2014), 2929-2962.

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