SCIENTIFIC BULLETIN
OF
THE POLITEHNICA UNIVERSITY OF TIMISOARA

Vol. 63(77), Issue 2 MATHEMATICS–PHISYCS 2018

A GENERALIZATION OF YOUNG'S THEOREM AND SOME APPLICATIONS

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Abstract

A generalization of classical Young's inequality for non-convex linear combinations is given, followed by applications to functionals. [1]

Keywords and phrases: *Young's inequality, isotonic linear functional*

1 Introduction

William Henry Young published in 1912 an inequality which extends the well known relation between arithmetic and geometric means. Now, that is called Young's inequality:

$$x^{\alpha}y^{\beta} \leq \alpha x + \beta y,$$

for any $x, y \geq 0$ and any positive α, β such that $\alpha + \beta = 1$.

In the last years, Young's inequality reappeared as a research theme and many improved inequalities, originated from that, were published by authors as: T. Ando, F. Kittaneh and Y. Manasrah, M. Tominaga, S. Furuichi, N. Minculete, J. M. Aldaz, S. S. Dragomir and O Hirzallah, see [2, 13, 14, 5, 10, 9, 11, 17, 1, 8, 7] and the references therein. T. Ando, O. Hirzallah and F. Kittaneh and Y. Manasrah used it for matrices and also S. Furuichi and N. Minculete and S. S. Dragomir used it for operators. Also W. Liao, J. Wu and J. Zhao and S. Manjeani generalized this inequality in recent years.

As a common feature of these new inequalities is the relation $\alpha + \beta = 1$.

In the followings we are going to state and prove inequalities beyond that condition.

[1]MSC (2010): 26D15

2 Generalization of Young's theorem

Theorem 2.1 *(a) In case* $\alpha + \beta > 1$, *and* $\alpha \in (0,1)$, *then*

$$\alpha x + \beta y > x^{\alpha} y^{\beta},$$

for all $x, y > 0$.

(b) In case $\alpha + \beta = 1$, *and* $\alpha, \beta \geq 0$, *then*

$$\alpha x + \beta y \geq x^{\alpha} y^{\beta},$$

for all $x, y \geq 0$.

(c) In case $\alpha < 0$, $\beta < 0$, *then*

$$\alpha x + \beta y < x^{\alpha} y^{\beta},$$

for all $x, y > 0$.

Proof. Let's find the extremes of the mapping

$$f(x,y) = \alpha x + \beta y - x^{\alpha} y^{\beta},$$

for $x, y > 0$, $\alpha, \beta \in \mathbf{R}_*$.

The stationary points of f are given by the system

$$\begin{cases} \frac{\partial f}{\partial x} = \alpha - \alpha x^{\alpha-1} y^{\beta} = 0 \\ \frac{\partial f}{\partial y} = \beta - \beta x^{\alpha} y^{\beta-1} = 0 \end{cases}.$$

That system is equivalent to the following one:

$$\begin{cases} x^{\alpha-1} y^{\beta} = 1 \\ x^{\alpha} y^{\beta-1} = 1 \end{cases}$$

which gives, by division: $\frac{y}{x} = 1$, hence $x = y$, and the unique stationary point of f is $(1,1)$. The hessian matrix of f is:

$$(Hf)(x,y) = -\begin{pmatrix} \alpha(\alpha-1)x^{\alpha-2}y^{\beta} & \alpha\beta x^{\alpha-1}y^{\beta-1} \\ \alpha\beta x^{\alpha-1}y^{\beta-1} & \beta(\beta-1)x^{\alpha}y^{\beta-2} \end{pmatrix} =$$

$$= -x^{\alpha-2}y^{\beta-2}\begin{pmatrix} \alpha(\alpha-1)y^2 & \alpha\beta xy \\ \alpha\beta xy & \beta(\beta-1)x^2 \end{pmatrix}.$$

The minor determinants Δ_1, Δ_2, of Sylvester's theorem, have the same sign as $d_1 = -\alpha(\alpha-1)$, $d_2 = -\left[\alpha\beta(\alpha-1)(\beta-1) - \alpha^2\beta^2\right] = -\alpha\beta(1-\alpha-\beta)$.

1. The point $(1,1)$ is a global minimum for f if $\Delta_1,\ \Delta_2 > 0,\ \forall\, x,\ y > 0$, that is $d_1, d_2 > 0$ or

$$\begin{cases} \alpha(\alpha-1) < 0 \\ \alpha\beta(1-\alpha-\beta) < 0 \end{cases} . \tag{1}$$

1.1 By the first inequality of (1), if $\alpha > 0$, then $\alpha - 1 < 0,\ \alpha < 1$, hence $\alpha \in (0,1)$. Here, we may have two cases, depending on the second inequality of (1):

1.1.1 If $\beta > 0$, then $1-\alpha-\beta < 0,\ \alpha+\beta > 1$, and

$$f(x,y) \geq f(1,1) = \alpha+\beta-1 > 0,\ \forall\, x,y > 0,$$

hence $f(x,y) > 0$ or $\alpha x + \beta y > x^\alpha y^\beta$, which is the statement (a).

1.1.2 If $\beta < 0$, then $1-\alpha-\beta > 0,\ \alpha+\beta < 1$, and

$$f(x,y) \geq f(1,1) = \alpha+\beta-1 < 0,\ \forall\, x,y > 0.$$

By that we have no conclusion.

1.2 If $\alpha < 0$, then $\alpha - 1 > 0,\ \alpha > 1$, and that is impossible.

2. The point (1,1) is a global maximum for f if $\Delta_1,\ \Delta_2 < 0,\ \forall\, x,\ y > 0$, that is $d_1, d_2 < 0$, or

$$\begin{cases} \alpha(\alpha-1) > 0 \\ \alpha\beta(1-\alpha-\beta) > 0 \end{cases} . \tag{2}$$

2.1 If $\alpha > 0$, then $\alpha > 1$, hence $\alpha > 1$.

2.1.1 If $\beta > 0$, then $1-\alpha-\beta > 0,\ \alpha+\beta < 1$, but these three conditions are incompatible.

2.1.2 If $\beta < 0$, then $1-\alpha-\beta < 0,\ \alpha+\beta > 1$, and

$$f(x,y) \leq f(1,1) = \alpha+\beta-1 > 0,\ \forall\, x,y > 0,$$

which gives no conclusion.

2.2 If $\alpha < 0$, then $\alpha - 1 < 0,\ \alpha < 1$, hence it remains that $\alpha < 0$.

2.2.1 If $\beta > 0$, then $1-\alpha-\beta < 0,\ \alpha+\beta > 1$, and

$$f(x,y) \leq f(1,1) = \alpha+\beta-1 > 0,\ \forall\, x,y > 0,$$

and we have no conclusion.

2.2.2 If $\beta < 0$, then $1 - \alpha - \beta > 0,\ \alpha + \beta < 1$, and

$$f(x, y) \leq f(1, 1) = \alpha + \beta - 1 < 0,\ \forall\, x, y > 0,$$

hence $f(x, y) < 0,\ \forall\ x, y > 0$, which is equivalent to the statement (c). The statement (b) is the classical Young's inequality.

In order to extend the previous Theorem 2.1, in the frame of functionals theory, we recall the following definition (one also may see [3], [4], [5]).

Definition 2.2 *Let E be a nonempty set and L be a linear class of real-valued functions $f, g : E \to \mathbf{R}$ having the following properties:*
(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha,\ \beta \in \mathbf{R}$.
(L2) $1 \in L$, i.e., if $f_0(t) = 1,\ \forall\, t \in E$, then $f_0 \in L$.
An isotonic linear functional is a functional $A : L \to \mathbf{R}$ having the following properties:
(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $\alpha,\ \beta \in \mathbf{R}$;
(A2) If $f \in L$ and $f(t) \geq 0$ then $A(f) \geq 0$.
The mapping A is said to be normalized if
(A3) $A(\mathbf{1}) = 1$.

The extension of the inequality (a) of Theorem 2.1 is stated as follows:

Theorem 2.2 *Let $A : L \to \mathbf{R}$ be an normalized isotonic linear functional. If $f, g \geq 0,\ f^{\alpha} g^{\beta} \in L$ and $A(f), A(g) > 0$ and α, β are real numbers so that $\alpha + \beta > 1,\ \alpha \in (0, 1)$ then the following inequality holds:*

$$(\alpha + \beta) A^{\alpha}(f) A^{\beta}(g) > A(f^{\alpha} g^{\beta}). \tag{3}$$

Now, if $f, g \geq 0,\ f^{\alpha} g^{\beta} \in L$ and $A(f), A(g) > 0$ and $\alpha < 0,\ \beta < 0$, then

$$(\alpha + \beta) A^{\alpha}(f) A^{\beta}(g) < A(f^{\alpha} g^{\beta}), \tag{4}$$

where $f, g : E \to \mathbf{R}$ are previous functions.

Proof. If we take in Theorem 2.1 (a), $x = \frac{f}{A(f)},\ y = \frac{g}{A(g)}$ then we get,

$$\alpha \frac{f}{A(f)} + \beta \frac{g}{A(g)} > \frac{f^{\alpha}}{A^{\alpha}(f)} \frac{g^{\beta}}{A^{\beta}(g)}.$$

Now, if we take the functional A in previous inequality, we find that

$$A\left(\alpha \frac{f}{A(f)} + \beta \frac{g}{A(g)}\right) > A\left(\frac{f^{\alpha}}{A^{\alpha}(f)} \frac{g^{\beta}}{A^{\beta}(g)}\right),$$

or

$$\alpha + \beta > \frac{A(f^{\alpha} g^{\beta})}{A^{\alpha}(f) A^{\beta}(g)},$$

or

$$(\alpha + \beta) A^{\alpha}(f) A^{\beta}(g) > A(f^{\alpha} g^{\beta}),$$

if $\alpha + \beta > 1,\ \alpha \in (0, 1)$ and $\beta > 0$.

For the second inequality, (4) , we consider Theorem 2.1, (c) and we put $x = \frac{f}{A(f)},\ y = \frac{g}{A(g)}$. Then we have,

$$\alpha \frac{f}{A(f)} + \beta \frac{g}{A(g)} < \frac{f^{\alpha}}{A^{\alpha}(f)} \frac{g^{\beta}}{A^{\beta}(g)},$$

and from here, using the functional A, we obtain,

$$\alpha + \beta < \frac{A(f^{\alpha} g^{\beta})}{A^{\alpha}(f) A^{\beta}(g)},$$

or

$$(\alpha + \beta) A^{\alpha}(f) A^{\beta}(g) < A(f^{\alpha} g^{\beta}),$$

where $\alpha < 0,\ \beta < 0$.

Another extension of the inequality (a) of the Theorem 2.1 is the following:

Theorem 2.3 *Let $A, B : L \to \mathbf{R}$ be two normalized isotonic linear functionals. If $f, g : E \to \mathbf{R}$ are so that $f \geq 0,\ g > 0,\ f^{\alpha} g^{1-\alpha},\ f^{\beta} g^{1-\beta} \in L$ and $\alpha, \beta \in \mathbf{R}$ with $\alpha + \beta > 1,\ \alpha \in (0, 1),\ \beta \in (0, 1),$ then we have:*

$$\alpha A(f) B(g) + \beta A(g) B(f) > A(f^{\alpha} g^{1-\alpha}) B(f^{\beta} g^{1-\beta}). \tag{5}$$

Proof. We use inequality (a) from Theorem 2.1 for $x = \frac{f(z)}{g(z)},\ y = \frac{f(t)}{g(t)}$, and we have:

$$\alpha \frac{f(z)}{g(z)} + \beta \frac{f(t)}{g(t)} > \frac{f^{\alpha}(z)}{g^{\alpha}(z)} \frac{f^{\beta}(t)}{g^{\beta}(t)}.$$

Multiplying by $g(z) g(t) > 0$ we obtain,

$$\alpha f(z) g(t) + \beta f(t) g(z) > f^{\alpha}(z) g^{1-\alpha}(z) f^{\beta}(t) g^{1-\beta}(t)$$

for any $z,\ t \in E$.

Fix $t \in E$ and then by previous inequality we have in the order of L that

$$\alpha f g(t) + \beta f(t) g > f^{\alpha} g^{1-\alpha} f^{\beta}(t) g^{1-\beta}(t).$$

If we take now the functional A in previous inequality then we have:

$$\alpha g(t) A(f) + \beta f(t) A(g) > f^{\beta}(t) g^{1-\beta}(t) A(f^{\alpha} g^{1-\alpha}),$$

for any $t \in E$.

This inequality can be written in the sense of the order of L as

$$\alpha g A(f) + \beta f A(g) > f^{\beta} g^{1-\beta} A(f^{\alpha} g^{1-\alpha}),$$

and now, if we take into account the functional B in last inequality, then we obtain the desired result.

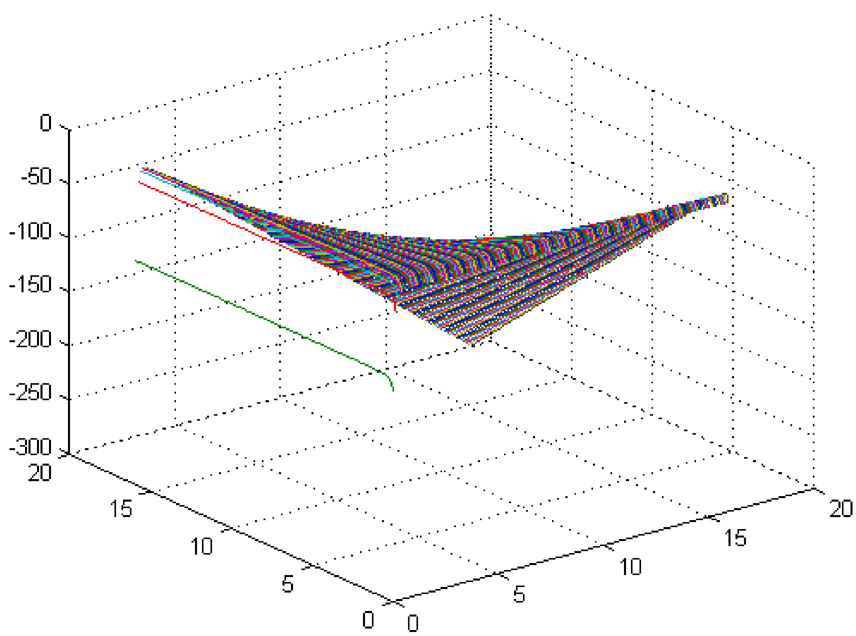

Figure 1: The graph of the function $f(x,y) = \alpha x + \beta y - x^{\alpha} y^{\beta}$ for $\alpha = -\frac{3}{7}$ and $\beta = -\frac{6}{7}$

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OF
THE POLITEHNICA UNIVERSITY OF TIMISOARA

Vol. 63(77), Issue 2 MATHEMATICS–PHISYCS 2018

APPLICATIONS OF THE NON-CONVEX YOUNG'S INEQUALITY IN HILBERT SPACES

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Abstract

A generalization of classical Young's inequality is applied for operators in Hilbert spaces. [1]

Keywords and phrases: *Young's inequality, operators, separable Hilbert spaces.*

1 Introduction

Let $\mathcal{B}(\mathcal{H})$ be the C^*- algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$, and let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive operators.

We recall the definition of the weighted arithmetic mean of A and B denoted by $A\nabla_\nu B$:

$$A\nabla_\nu B = (1-\nu)A + \nu B,$$

where $\nu \in [0,1]$.

If A is invertible then the weighted geometric mean of A and B, denoted by $A\sharp_\nu B$, is defined by:

$$A\sharp_\nu B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^\nu A^{\frac{1}{2}}.$$

When $\nu = \frac{1}{2}$ we notate $A\nabla B$ and $A\sharp B$ instead of $A\nabla_{\frac{1}{2}} B$ and $A\sharp_{\frac{1}{2}} B$.

If A and B are positive invertible operators, it is well-known that:

$$A\nabla_\nu B \geq A\sharp_\nu B, \forall\, \nu \in (0,1),$$

which is the operatorial version of the classical Young's inequality, see [9]

In the followings we will give some variants of the non-convex Young's operatorial inequality based on [10].

[1]MSC (2010): 26D15

2 Main results

Proposition 2.1 *Let A, B be two positive invertible operators on $\mathcal{H}$ so that there is an $r > 0$ such*

$$(1-r)B \leq A \leq (1+r)B.$$

Then we have:

$$\alpha A + \beta B > B\sharp_\alpha A, \tag{1}$$

for any $\alpha,\ \beta$ fulfilling the conditions $\alpha+\beta > 1,\ \alpha \in (0,1)$.
Moreover,

$$\alpha A + \beta A^\beta > A^{\alpha+\beta^2}, \tag{2}$$

as well as

$$\alpha A + \beta A^\alpha > A^{\alpha+\alpha\beta}, \tag{3}$$

for all $\alpha,\ \beta$ checking $\alpha+\beta > 1$ and $\alpha \in (0,1)$.

Proof. We take $y = 1$ in the inequality (a) of Theorem 2.1 presented in [10] and we get $\alpha x + \beta y > x^\alpha$ when $\alpha + \beta > 1$ and $\alpha \in (0,1)$.

Using the functional calculus with continuous functions of spectrum, see [1] page 8, we find out that

$$\alpha X + \beta I > X^\alpha,$$

where X is the strictly positive operator on $\mathcal{H}$.

If we put instead of the operator X the strictly positive operator $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ we obtain

$$\alpha B^{-\frac{1}{2}}AB^{-\frac{1}{2}} + \beta I > \left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^\alpha,$$

when $\alpha+\beta > 1$ and $\alpha \in (0,1)$.

Multiplying both sides of previous inequality by $B^{\frac{1}{2}}$, it results

$$\alpha A + \beta B > B^{\frac{1}{2}}\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^\alpha B^{\frac{1}{2}},$$

when $\alpha+\beta > 1$ and $\alpha \in (0,1)$, which is the relation (1) of the statement.

For the second inequality considering $y = x^\beta > 0$ and then $y = x^\alpha > 0$ in the inequality (a) of Theorem 2.1 ([10]) we obtain

$$\alpha x + \beta x^\beta > x^\alpha y^{\beta^2},\ \alpha x + \beta x^\alpha > x^{\alpha+\alpha\beta}$$

for any $\alpha,\ \beta$ fulfilling $\alpha+\beta>1$ and $\alpha\in(0,1)$.

Using again the fumctional calculus with continuous functions on spectrum, for the strictly positive operator A, we have

the relations (2) and (3) of the statement.

Proposition 2.2 *Let $X,\ Y$ be two strictly positive operators on $\mathcal{H}$, then there is $r>0$ having the properties $(1-r)I\leq X,Y\leq(1+r)I$, such that for any $\alpha,\ \beta\in\mathbf{R}$ $\alpha+\beta>1$ and $\alpha\in(0,1)$, it is true that*

$$\alpha X+\beta Y+(\alpha+\beta)I\geq X^{\frac{\alpha}{2}}Y^{\frac{\beta}{2}}+Y^{\frac{\beta}{2}}X^{\frac{\alpha}{2}}. \tag{4}$$

Proof. We know by Theorem 2.1 (a) ([10]) that there is $r>0$, such that

$$\alpha x+\beta y\geq x^{\alpha}y^{\beta}$$

for any $x,y\in[1-r.1+r]$, when $\alpha+\beta>1$ and $\alpha\in(0,1)$,

In particular, for $x=1$, it results that

$$\alpha+\beta y\geq y^{\beta}$$

and for $y=1$, it results that

$$\alpha x+\beta\geq x^{\alpha}$$

in the same mentioned conditions.

Using now the functional calculus with continuous functions on the spectrum we will respectivelly find

$$\alpha I+\beta Y\geq Y^{\beta}$$

and

$$\alpha X+\beta I\geq X^{\alpha}.$$

Then

$$\alpha X+\beta Y+(\alpha+\beta)I\geq X^{\alpha}+Y^{\beta} \tag{5}$$

For any strictly positive operators U,V it is known that $(U-V)^2\geq 0$, hence $U^2+V^2\geq UV+VU$.

By that result applied in (5) taking $U=X^{\frac{\alpha}{2}}$ and $V=Y^{\frac{\beta}{2}}$ it results the desired inequality (4) of the statement.

Corollary 2.3 *In particular, if* $X = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$, $Y = B^{-\frac{1}{2}}CB^{-\frac{1}{2}}$, *(where* A, B, C *are strictly positive operators) check the hypothesis of Proposition 2.2 then*

$$\alpha A + \beta C + (\alpha+\beta)B >$$

$$> B^{\frac{1}{2}}\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^{\frac{\alpha}{2}}\left(B^{-\frac{1}{2}}CB^{-\frac{1}{2}}\right)^{\frac{\beta}{2}}B^{\frac{1}{2}}+B^{\frac{1}{2}}\left(B^{-\frac{1}{2}}CB^{-\frac{1}{2}}\right)^{\frac{\beta}{2}}\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^{\frac{\alpha}{2}}B^{\frac{1}{2}}.$$

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