SCIENTIFIC BULLETIN
OF
THE POLITEHNICA UNIVERSITY TIMISOARA

Vol 63(77), Issue 1 MATHEMATICS –PHYSICS 2018

DIRECT LIMIT OF MATRIX-RINGS MAY BE UNITAL

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Abstract

Although the matrix-rings *M(m;R)* (R is a unital commutative ring) are unital rings, yet their classical direct limit is a non-unital ring. It is presented a direct system of matrix-rings that has a unital ring as a direct limit. [1]

Keywords and phrases: *block-diagonal matrix, direct limit, direct system, matrix-ring.*

1 Introduction

Let $\mathbf{N} = \{1, 2, \ldots\}$, let R be a unital commutative ring of characteristic zero, and let *Rng*, *Ring*, *R-mod* respectively be the classical categories: rings, unital rings and unital morphism, R-modules.

It is known that the direct limit, in *Rng*, of the matrix rings *M(m;R)*, $m \in \mathbf{N}$ is isomorph to $R^{(\mathbf{N}\times\mathbf{N})}$, which belongs to *Rng*, although the matrix-rings belong to *Ring*. By $R^{(\mathbf{N}\times\mathbf{N})}$ we mean the R-module of all mappings $f : \mathbf{N} \times \mathbf{N} \longrightarrow R$ having finite support (only a finite number of images are non-zero). Any such mapping may be considered as a double infinite matrix

$$M = \begin{pmatrix} f(1,1) & f(1,2) & \dots \\ f(2,1) & f(2,2) & \dots \\ \vdots & & \end{pmatrix}$$

Thus, one may say that the direct limit of the matrix-ring in *Rng* is the set of infinite matrices having a finite support with the usual operations extended as much as needed. That renders $R^{(\mathbf{N}\times\mathbf{N})}$ as a non-unital ring. We are going to use a special case of block-diagonal matrices:

1. diag(A;r):= diag(A,...,A), where A appears on r slots

[1]MSC (2010): 13A99

2. diag(A,∞):=diag(A,...,A,...)$\in R^{\mathbf{N}\times\mathbf{N}}$

The only possible unit of $R^{\mathbf{N}\times\mathbf{N}}$ is diag(1,∞) , which has no finite support, hence $R^{(\mathbf{N}\times\mathbf{N})} \in$ *Rng−Ring*.

Is it possible to have a unital ring as the direct limit of matrix-rings ? The answer will be given in the followings. Firstly, one must remark that the direct system of matrix-rings used for the direct limit in *Rng* is not a direct system in *Ring*. Indeed, the mappings of the usual direct system are

$$f_{mn} : M(m;R) \longrightarrow M(n;R)$$

for any $m < n,\ f_{mn}(M) = (\nu_{ij})$, where

$$\nu_{ij} = \begin{cases} \mu_{ij},\ \ i,j \le n \\ 0,\ \ else \end{cases}$$

and $M = (\mu_{ij})$. Those are not unital morphisms ($f_{mn}(I_m) \neq I_n$) In fact, the direct limit in *Rng* is the direct limit in the category *R-mod*, plus the remark that the objects and the morphisms implied belong to the category *Rng*, see [1], p. 34. Therefore, if one wants to have a direct limit in *Ring*, one must firstly find a direct system of matrix-rings in *Ring*. But here there is a problem shown in the following theorem.

Theorem 1.1 *If $f : M(m;R) \longrightarrow M(n;R)$ is a* Ring-*morphism for $m < n$, then $m|n$ (m divides n).*

Proof. For the properties of matrices mentioned here, one may see [2] or the Romanian translation [3]. Lets suppose $f : M(m;R) \longrightarrow M(n;R)$ is a *Ring*-morphism. That means:

1. $f(M_1 + M_2) = F(M_1) + f(M_2),\ \forall M_1, M_2 \in M(m;R)$
2. $f(M_1 \cdot M_2) = f(M_1) \cdot f(M_2),\ \forall M_1, M_2 \in M(m;R)$
3. $f(I_m) = I_n$.

The identity matrix I_m may be decomposed into a sum

$$I_m = \sum_{i=1}^{m} E_i$$

where $E_i = (\delta_j^i \cdot \delta_k^i)_{jk}$

is the matrix having just one non-zero entry in the cell
(i, i). The images of E_i, denoted by $F_i = f(E_i)$, $i = 1, 2, \ldots .m$, inherit properties of E_i, for example:

a) $\sum_{i=1}^{m} F_i = I_n$

b) the ranks of F_i are all equal.

The a) statement is due to 1) and 3). For b) we consider the matrix M_k, which is obtain from I_m by exchanging the k-th and the (k+1)-th rows, that is

$$M_k = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 & \ldots & 0 \\ \vdots & & & & & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}$$

The effect of leftwards multiplication of any matrix $M \in M(m; R)$ by M_k is the exchange of the k-th row by the (k+1)-th row in M. The effect of rightwards multiplication is the exchange of the k-th by the (k+1)-th column in M. All the matrices M_k, $k = 1, \ldots, m-1$ are invertible, since their determinants equal
-1, the opposite of 1 in R, which is invertible in R. Then the images by f of M_k, denoted by $N_k = f(M_k)$, are invertible due to 2.) and 3.). By repeated multiplication by M_k, leftwards and rightwards, it is possible to connect any two matrices E_i. Then, any two matrices F_i may be connected by leftwards and rightwards multiplication using invertible matrices in *M(n, R)*, as a consequence of 2.) and 3.). That means F_i has the same rank, $\forall i \in \{1, \ldots, m\}$.

On the other hand for any $B \in M(n; R)$ there is a unique decomposition of B in terms of $F_i : B = \sum_{i=1}^{m} BF_i$, by multiplying the relation a.) by B. Indeed, if $\sum BF_i = \sum CF_i$, it results that $\sum (B - C)F_i = 0$, hence $B - C = 0$. That means BF_i, BF_j have non-zero entries in different cells $\forall i, j$. In order to realise that F_i must have zero-columns. The same is true for F_iB, but here F_i must have zero-rows. Further, all the F_i must have the same number of zero-rows (and zero-columns), else their ranks wouldn't equal. It results that $m|n$.

Corollary 1.2 *The conclusion is that there is no direct system in* Ring *made by the matrix-rings* M(m; R)*, over the index set* ***N****, endowed by the usual order relation.*

Still, we may have a direct system of matrix-rings over **N**, as is stated in the followings.

Lemma 1.3 *The family* M(m; R)*,* $m \in \mathbf{N}$ *is a direct system in* Ring*, over the index set* $\mathbf{N}$*, endowed by the relation "divides".*

Proof The set $\mathbf{N}$ and the "divides" relation is a directed set. If $m|n$, there is $r \in \mathbf{N}$ such that $n = r \cdot m$. Then the *Ring*-morphism

$$f_{mn} : M(m;R) \longrightarrow M(n;R),\ f_{mn}(M) = diag(M, r)$$

is the ring-morphism of a direct system in *Ring*. The requirements of direct system are:

1. $f_{mm} = id,\ \forall m \in \mathbf{N}$ (obvious)
2. $f_{np} \circ f_{mn} = f_{mp},\ \forall m|n|p.$

Indeed, supposing that $n = r \cdot m,\ p = s \cdot n$, we have $f_{mn}(M) = diag(M, r)$, then $f_{np}(f_{mn}(M)) = f_{np}(diag(M, r)) = diag(M, rs)$, and also $f_{mp}(M) = diag(M, rs)$.

Theorem 1.4 *The direct limit in* Ring *of the matrix-rings* $M(m;R),\ m \in \mathbf{N}$*, corresponding to the direct system of the Lemma 1.3, is:*

$$L = \{diag(M, \infty)\ |M \in M(m;R), \forall m \in \mathbf{N}\}.$$

Proof. Following [1], p. 33, the direct limit of the direct system in Lemma 1.3 is constructed by considering the direct sum of the R-modules $M(m;R),\ m \in \mathbf{N}$. Those are identified by their images in the direct sum. The direct limit is the quotient set of the direct sum by the R-submodule generated by all the elements $M - f_{mn}(M), \forall M \in M(m;R)$ and $m|n$. That is, the image in the direct sum of any $M \in M(m;R)$ is identified by its image $f_{mn}(M) = diag(M, r)$, where $n = rm$. Hence all the diagonal matrices *diag (M, r)* are identified, and the equivalence class bijectively corresponds to $diag(M, \infty)$. Hence the quotient set is L. L is also a unital ring, and all the implied morphisms are morphisms of unital rings. That results by [1], p. 34 or by straightforward computation.

References

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Vol 63(77), Issue 1 MATHEMATICS –PHYSICS 2018

SOME NEW REMARKS ON THE FALKNER-SKAN EQUATION: STABILIZATION, INSTABILITY AND LAX FORMULATION

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Abstract

In this paper we study the Falkner-Skan equation. Some stability problems, Lax formulation and an approximate analytic solution by means of the Optimal Homotopy Asymptotic Method (OHAM) were discussed. [1]

Keywords and phrases: *stability, Lax formulation, optimal homotopy asymptotic method (OHAM), nonlinear differential system.*

1 Introduction

The proprieties of viscoelastic materials have been intensively studied in recent years because of their industrial and technological applications such as plastic processing, cosmetics, paint flow, adhesives, accelerators, electrostatic filters, etc [1].

The Falkner-Skan equation describing this proprieties were studied from various points of view: some approximate procedures to solve a boundary layer equations [2], numerical solution [3], existence of a unique smooth solution [4], [5] and [6], was analytically investigated [7] and [8], by using Adomian decomposition method [9] and [10], etc.

The aim of the present paper is to propose a geometrical point of view and an accurate approach to Falkner-Skan equation using an analytical technique, namely optimal homotopy asymptotic method [11], [12], [13].

The validity of our procedure, which does not imply the presence of a small parameter in the equation, is based on the construction and determination of the auxiliary functions combined with a convenient way to optimally control the

[1]MSC (2010): 34-XX, 34A26, 34H05, 34M45, 35A24, 37C10, 49J15, 49K15, 65Lxx, 93C15, 93D05

convergence of the solution. The efficiency of the proposed procedure is proves while an accurate solution is explicitly analytically obtained in an iterative way after only one iteration.

From the geometry point of view, we establish the equilibrium states of the studied system and define a control function. Using specific Hamilton-Poisson geometry methods, namely the energy-Casimir method [14] we are able to study the nonlinear stability of these equilibrium states.

In this paper, a control function is proposed in order to study the stability of the equilibrium states of the system and the numerical integration via the Optimal Homotopy Asymptotic Method of the controlled system is presented.

The paper is organized as follows: in the second paragraph we put the Falkner-Skan equation in a differential system form and find the equilibrium states of the system. In the third section we find a control which preserves the equilibrium states of the system and give a Hamilton-Poisson realization of a controlled system. The fourth section is dedicated to study of stability of the controlled system. In a fifth paragraph is given a Lax formulation for the controlled system and finally in the sixth section a briefly presentation of the Optimal Homotopy Asymptotic Method, developed in [13] and used in the last part in order to obtain the approximate analytic solutions of the controlled system.

2 The Falkner-Skan equation in the flow of a viscous fluid

The dimensionless Falkner-Skan equation in the flow of a viscous fluid can be written as [2], [3], [7], [10]:

$$X'''(t) + X(t)X''(t) + \beta\Big(1 - (X'(t))^2\Big) = 0, \tag{1}$$

with the initial and boundary conditions

$$X(0) = 0,\ X'(0) = 0,\ \lim_{t\to\infty} X'(t) = 1, \tag{2}$$

where $t > 0$, β is a measure of the pressure gradient, and prime denotes derivative with respect to t.

Using the notations:

$$X(t) = x_1(t), \quad X'(t) = x_2(t), \quad X''(t) = x_3(t),$$

the nonlinear equation Eq. (1) becomes:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = -\beta\Big(1 - x_2^2\Big) - x_1 x_3 \end{cases}, \qquad t > 0. \tag{3}$$

The nonlinear differential system (3) has an equilibrium state $e^M = (M, 0, 0), M \in \mathbf{R}$ iff $\beta = 0$.

3 The Hamilton-Poisson realization of the system (3)

For the beginning, let us recall very briefly the definitions of general Poisson manifolds and the Hamilton-Poisson systems.

Definition: Let M be a smooth manifold and let $C^\infty(M)$ denote the set of the smooth real functions on M. A **Poisson bracket on** M is a bilinear map from $C^\infty(M) \times C^\infty(M)$ into $C^\infty(M)$, denoted as:

$$(F, G) \mapsto \{F, G\} \in C^\infty(M), F, G \in C^\infty(M)$$

which verifies the following properties:
- skew-symmetry:

$$\{F, G\} = -\{G, F\};$$

- Jacobi identity:

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0;$$

- Leibniz rule:

$$\{F, G \cdot H\} = \{F, G\} \cdot H + G \cdot \{F, H\}.$$

Proposition: Let $\{\cdot, \cdot\}$ a Poisson structure on $\mathbf{R}^n$. Then for any $f, g \in C^\infty(\mathbf{R}^n, \mathbf{R})$ the following relation holds:

$$\{f, g\} = \sum_{i,j=1}^{n} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Let the matrix given by:

$$\Pi = [\{x_i, x_j\}].$$

Proposition: Any Poisson structure $\{\cdot, \cdot\}$ on $\mathbf{R}^n$ is completely determined by the matrix Π via the relation:

$$\{f, g\} = (\bigtriangledown f)^t \Pi (\bigtriangledown g).$$

Definition: A **Hamilton-Poisson system** on $\mathbf{R}^n$ is the triple $(\mathbf{R}^n, \{\cdot,\cdot\}, H)$, where $\{\cdot,\cdot\}$ is a Poisson bracket on $\mathbf{R}^n$ and $H \in C^\infty(\mathbf{R}^n, \mathbf{R})$ is the energy (Hamiltonian). Its dynamics is described by the following differential equations system:

$$\dot{x} = \Pi \cdot \nabla H$$

where $x = (x_1, x_2, ...x_n)^t$.

Definition: Let $\{\cdot,\cdot\}$ a Poisson structure on $\mathbf{R}^n$. A **Casimir** of the configuration $(\mathbf{R}^n, \{\cdot,\cdot\})$ is a smooth function $C \in C^\infty(\mathbf{R}^n, \mathbf{R})$ which satisfy:

$$\{f, C\} = 0, \forall f \in C^\infty(\mathbf{R}^n, \mathbf{R}).$$

Let us employ the control $u \in C^\infty(\mathbf{R}^3, \mathbf{R})$,

$$u(x_1, x_2, x_3) = (0\ ,\ x_1x_2\ ,\ -x_2^2 - x_1^2x_2), \tag{4}$$

for the system (3). The controlled system (3)$-$(4), explicitly given by:

$$\left\{\begin{array}{l} x_1' = x_2 \\ x_2' = x_3 + x_1x_2 \qquad , \\ x_3' = -x_1x_3 - x_2^2 - x_1^2x_2. \end{array}\right. \qquad t > 0, \tag{5}$$

Proposition: The controlled system (5) has the Hamilton-Poisson realization

$$(\mathbf{R}^3, \Pi_-, H),$$

where

$$\Pi_- = \begin{bmatrix} 0 & 1 & -x_1 \\ -1 & 0 & x_2 \\ x_1 & -x_2 & 0 \end{bmatrix}$$

is the minus Lie-Poisson structure and

$$H(x_1, x_2, x_3) = \frac{1}{2}x_2^2 - x_1x_3 - x_1^2x_2$$

is the Hamiltonian.

Proof: Indeed, we have:

$$\Pi_- \cdot \nabla H = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

and the matrix Π_- is a Poisson matrix, see [15].

The next step is to find the Casimirs of the configuration described by the above Proposition. Since the Poisson structure is degenerate, there exist Casimir functions. The defining equations for the Casimir functions, denoted by C, are

$$\Pi^{ij}\partial_j C = 0.$$

It is easy to see that there exists only one functionally independent Casimir of our Poisson configuration, given by $C : \mathbf{R}^3 \to \mathbf{R}$,

$$C(x_1, x_2, x_3) = -x_3 - x_1 x_2.$$

Consequently, the phase curves of the dynamics Eq. (5) are the intersections of the surfaces $H(x_1, x_2, x_3) = const.$ and $C(x_1, x_2, x_3) = const.$.

4 Stability Problem

The concept of stability is an important issue for any differential equation. The nonlinear stability of the equilibrium point of a dynamical system can be studied using the tools of mechanical geometry, so this is another good reason to find a Hamilton -Poisson realization. For more details, see [15]. We start this section with a short review of the most important notions.

Definition: An equilibrium state x_e is said to be **nonlinear stable** if for each neighbourhood U of x_e in D there is a neighbourhood V of x_e in U such that trajectory $x(t)$ initially in V never leaves U.

This definition supposes well-defined dynamics and a specified topology. In terms of a norm $\| \, \|$, nonlinear stability means that for each $\varepsilon > 0$ there is $\delta > 0$ such that if

$$\|x(0) - x_e\| < \delta$$

then

$$\|x(t) - x_e\| < \varepsilon, \quad (\forall) \quad t > 0.$$

It is clear that nonlinear stability implies spectral stability; the converse is not always true.

The equilibrium states of the dynamics Eq. (1) are

$$e^M = (M, 0, 0), \quad M \in \mathbf{R}.$$

Proposition 1: For the equilibrium states $e^M = (M, 0, 0)$ the following statements hold:

a) $e^M = (M, 0, 0)$ are unstable for $M > 0$;
b) $e^M = (M, 0, 0)$ are unstable for $M = 0$

Proof: We will use energy-Casimir method, see [15] for details. Let

$$F_\varphi(x_1, x_2, x_3) = H(x_1, x_2, x_3) + \varphi[C(x_1, x_2, x_3)] =$$

$$= \frac{1}{2}x_2^2 - x_1x_3 - x_1^2x_2 + \varphi(-x_3 - x_1x_2)$$

be the energy-Casimir function, where $\varphi : \mathbf{R} \to \mathbf{R}$ is a smooth real valued function.

Now, the first variation of F_φ is given by

$$\delta F_\varphi(x_1, x_2, x_3) = x_2\delta x_2 - x_1\delta x_3 - x_3\delta x_1 - 2x_1x_2\delta x_1 - x_1^2\delta x_2 +$$

$$+\dot{\varphi}\,(-x_3 - x_1x_2) \cdot (-x_1\delta x_2 - x_2\delta x_1 - \delta x_3)$$

so we obtain

$$\delta F_\varphi(e^M) = [M + \dot{\varphi}\,(0)] \cdot (-M\delta x_2 - \delta x_3)$$

that is equals zero for any $M \in \mathbf{R}^*$ if and only if

$$\dot{\varphi}\,(0) = -M. \tag{6}$$

The second variation of F_φ at the equilibrium of interest is given by

$$\delta^2 F_\varphi(e^M) = [\ddot{\varphi}(0)]^{-1} \cdot [\ddot{\varphi}(0)\delta x_3 - \delta x_1 + M \cdot \ddot{\varphi}(0)\delta x_2]^2 +$$

$$+[\ddot{\varphi}(0)]^{-1}[1 + M^2\ddot{\varphi}(0) - M^2(\ddot{\varphi}(0))^2]^{-1} \cdot \Big[\Big(1 + M^2\ddot{\varphi}(0) - M^2(\ddot{\varphi}(0))^2\Big)\delta x_2 +$$

$$+\Big(M\ddot{\varphi}(0) - M\Big)\delta x_1\Big]^2 + [1+M^2\ddot{\varphi}(0) - M^2(\ddot{\varphi}(0))^2]^{-1} \cdot [-1 + M^2\ddot{\varphi}(0) - M^2]\,\Big(\delta x_1\Big)^2.$$

If we choose now φ such that the relation (6) is valid and $\delta^2 F_\varphi(e^M)$ is positive defined, i.e.

$$\ddot{\varphi}(0) > 0 \;\text{ and }\; 1 + M^2\ddot{\varphi}(0) - M^2(\ddot{\varphi}(0))^2 > 0 \;\text{ and }\; -1 + M^2\ddot{\varphi}(0) - M^2 > 0$$

then the second variation of F_φ at the equilibrium of interest is positive defined.

We can assume that $M > 0$. From these inequalities we deduce that:

$$\frac{M^2 + 1}{M^2} < \varphi''(0) < \frac{M^2 + M\sqrt{M^2 + 4}}{2M^2},$$

that implies

$$2M^2 + 2 < M^2 + M\sqrt{M^2+4} \quad \Rightarrow \quad (M^2+2)^2 < M^2(M^2+4) \quad \Rightarrow \quad 4 < 0,$$

that is false.

Therefore, the equilibrium state $e^M(M,0,0)$ is unstable.

In the same way, we conclude that $e^M(M,0,0)$ is unstable for $M = 0$. □

Table 1: The comparison between the approximate solutions $\bar{x}_1$ given by Eq. (7) and the corresponding numerical solutions for $\beta = 0$
(relative errors: $\epsilon_{x_1} = |x1_{numerical} - \bar{x}_1|$)

t	$x1_{numerical}$	$\bar{x}_1$ given by Eq. (7)	ϵ_{x_1}
0	$1.5671 \cdot 10^{-25}$	$-1.3322 \cdot 10^{-15}$	$1.3322 \cdot 10^{-15}$
4/5	0.149674539444	0.149401535388	$2.73004 \cdot 10^{-4}$
8/5	0.582956328320	0.582978942361	$2.2614 \cdot 10^{-5}$
12/5	1.231527648000	1.231539489179	$1.1841 \cdot 10^{-5}$
16/5	1.990581010375	1.990607740256	$2.6729 \cdot 10^{-5}$
4	2.783886492275	2.783817337929	$6.9154 \cdot 10^{-5}$
24/5	3.583254092715	3.583303973631	$4.9880 \cdot 10^{-5}$
28/5	4.383220411026	4.383289581820	$6.9170 \cdot 10^{-5}$
32/5	5.183219409763	5.183234915896	$1.5506 \cdot 10^{-5}$
36/5	5.983219388168	5.983199995268	$1.9392 \cdot 10^{-5}$
8	6.783219382599	6.783194882759	$2.4499 \cdot 10^{-5}$

5 Lax formulation

Let introduce the matrices:

$$L = \begin{pmatrix} \frac{1}{2}x_2^2 & -x_1x_3 - x_3 - \frac{1}{8}x_2^4 - \frac{1}{2}(-x_1x_3 - x_1^2x_2)^2 \\ 1 & -x_1x_3 - x_1^2x_2 \end{pmatrix} ,$$

$$B = \begin{pmatrix} 1 & -x_2(x_3 + x_1x_2) \\ 0 & 1 \end{pmatrix}.$$

Then an easy computation we can establish the following result:

Theorem 5.1 *The controlled system (5) have a Lax formulation, i.e., it can be put in the equivalent form:*

$$\frac{dL}{dt} = \Big[L\ ,\ B\Big] \quad \Leftrightarrow \quad \frac{dL}{dt} = L \cdot B - B \cdot L.$$

As in [15], the following properties hold:

$$H = Trace(L) \quad \text{and} \quad C = \frac{1}{2}Trace(L^2),$$

where H- Hamiltonian function and C- Casimir function.

Table 2: The comparison between the approximate solutions $\bar{x}'_1$ from Eq. (7) and the corresponding numerical solutions for $\beta = 0$ (relative errors: $\epsilon_{x'_1} = |x'_{1_{numerical}} - \bar{x}'_1|$)

t	$x'_{1_{numerical}}$	$\bar{x}'_1$ from Eq. (7)	$\epsilon_{x'_1}$
0	-3.8645 $\cdot 10^{-21}$	8.8817 $\cdot 10^{-16}$	8.8818 $\cdot 10^{-16}$
4/5	0.371963259413	0.372477797312	5.1453 $\cdot 10^{-4}$
8/5	0.696699514599	0.696023892471	6.7562 $\cdot 10^{-4}$
12/5	0.901065461379	0.901471382767	4.0592 $\cdot 10^{-4}$
16/5	0.980364982283	0.980092859963	2.7212 $\cdot 10^{-4}$
4	0.997770087958	0.997861518336	9.1430 $\cdot 10^{-5}$
24/5	0.999859396033	0.999974757114	1.15361 $\cdot 10^{-4}$
28/5	0.999995149208	0.999946428194	4.8721 $\cdot 10^{-5}$
32/5	0.999999902864	0.999935247532	6.4655 $\cdot 10^{-5}$
36/5	0.999999992429	0.999977995910	2.1996 $\cdot 10^{-5}$
8	0.999999993273	1.000005054164	5.0608 $\cdot 10^{-6}$

6 Numerical simulation

In this section, the accuracy and validity of the OHAM technique is proved using a comparison of our approximate solutions with numerical results obtained via the fourth-order Runge-Kutta method for $\beta = 0$.

The convergence-control parameters $K, C_i, i = \overline{1,\ 8}$ are optimally determined by means of the least-square method using the Mathematica 9.0 software.

Observation: If $\bar{x}(t)$ is the approximate analytic solution obtained via Optimal Homotopy Asymptotic Method [13], then for $\beta = 0$ the convergence-control parameters are respectively :

$$C_1 = -5.146692834756\ ,\ C_2 = -3.319352427903\ ,\ C_3 = 1.365481026558\ ,$$

$$C_4 = -0.109053890316\ ,\ C_5 = 63.014570679440\ ,\ C_6 = -183.226725640327\ ,$$

$$C_7 = 47.317886321776\ ,\ C_8 = 53.798097627583\ ,\ K = 1.679601787261\ .$$

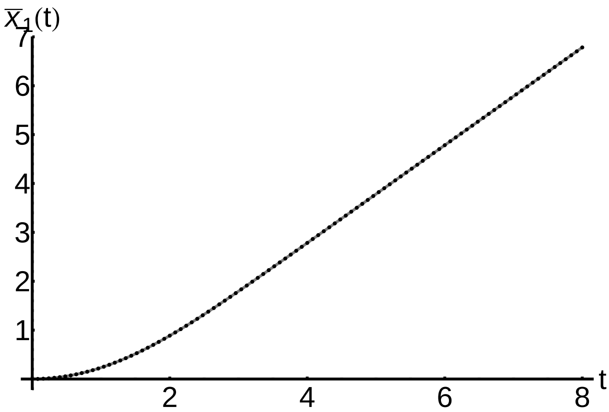

Figure 1: Comparison between the approximate solutions $\bar{x}_1$ given by Eq. (7) and the corresponding numerical solutions:
———— numerical solution, $\cdots\cdots\cdots$ OHAM solution.

Table 3: The comparison between the approximate solutions $\bar{x}_1''$ from Eq. (7) and the corresponding numerical solutions for $\beta = 0$ (relative errors: $\epsilon_{x_1''} = |x''_{1_{numerical}} - \bar{x}_1''|$)

t	$x''_{1_{numerical}}$	$\bar{x}_1''$ from Eq. (7)	$\epsilon_{x_1''}$
0	0.469599995897	0.469599895897	$1.0000 \cdot 10^{-7}$
4/5	0.451190185801	0.460093505720	$8.9033 \cdot 10^{-3}$
8/5	0.342486827279	0.342756074406	$2.6924 \cdot 10^{-4}$
12/5	0.167560529122	0.167075663254	$4.8486 \cdot 10^{-4}$
16/5	0.046370185755	0.046300824641	$6.9361 \cdot 10^{-5}$
4	0.006874039262	0.007295190897	$4.2115 \cdot 10^{-4}$
24/5	0.000538393988	0.000306981990	$2.3141 \cdot 10^{-4}$
28/5	0.000022211398	$-9.0012 \cdot 10^{-5}$	$1.1222 \cdot 10^{-4}$
32/5	$4.7872 \cdot 10^{-7}$	$4.2932 \cdot 10^{-5}$	$4.2454 \cdot 10^{-5}$
36/5	$4.0075 \cdot 10^{-9}$	$4.9247 \cdot 10^{-5}$	$4.9243 \cdot 10^{-5}$
8	$6.8538 \cdot 10^{-10}$	$1.8697 \cdot 10^{-5}$	$1.8696 \cdot 10^{-5}$

The first-order approximate solutions proposed in [13] becomes:

$$
\begin{aligned}
&\bar{x}_1(t) = -1.216776769791 - 0.236541146259 \cdot e^{-6.718407149045t} + t + \\
&+ e^{-5.038805361784t} \cdot (2.067180899886 + 0.318125112302t - 0.554796671887t^2) + \\
&+ e^{-3.359203574522t} \cdot (-4.570156113663 - 5.624329520194t - 8.333219504464t^2 - \\
&- 4.761785285840t^3) + e^{-1.679601787261t} \cdot (3.956293129828 + 4.426059140587t - \\
&- 1.432577756121t^2 - 0.407499225829t^3 + 0.162858423313t^4 - 0.012985684004t^5) \ .
\end{aligned} \tag{7}
$$

Finally, Tables 1 - 3 and Figs. 1-2 emphasize the accuracy of the OHAM technique by comparing the approximate analytic solutions $\bar{x}_1$, $\bar{x}_1'$ and $\bar{x}_1''$ respectively presented above with the corresponding numerical integration values.

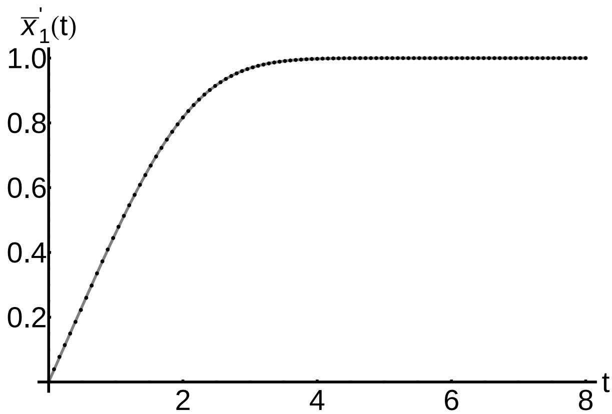

Figure 2: Comparison between the approximate solutions $\bar{x}_1'$ from Eq. (7) and the corresponding numerical solutions: ——— numerical solution, ··········· OHAM solution.

7 Conclusion

In this paper we analyze the Falkner-Skan equations from some geometrical point of view. The stability of a nonlinear differential problem governing the Falkner-Skan equation is investigated. Finding a Hamilton-Poisson realization, the results were obtained using specific tools, such as the energy-Casimir method. We give find a Lax formulation for the studied system.

Finally, the analytical integration of the nonlinear system (obtained via the Optimal Homotopy Asymptotic Method and presented in [13]) is compared with the exact solution (obtained as intersections of the surfaces $H(x_1, x_2, x_3) = const.$ and $C(x_1, x_2, x_3) = const$).

Numerical integration of the controlled dynamics is obtained via the Optimal Homotopy Asymptotic Method. Numerical simulations and a comparison with Runge-Kutta 4 steps integrator are presented, too.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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