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SOME NEW REMARKS ON THE FALKNER-SKAN EQUATION: STABILIZATION, INSTABILITY AND LAX FORMULATION

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Abstract

In this paper we study the Falkner-Skan equation. Some stability problems, Lax formulation and an approximate analytic solution by means of the Optimal Homotopy Asymptotic Method (OHAM) were discussed. [1]

Keywords and phrases: *stability, Lax formulation, optimal homotopy asymptotic method (OHAM), nonlinear differential system.*

1 Introduction

The proprieties of viscoelastic materials have been intensively studied in recent years because of their industrial and technological applications such as plastic processing, cosmetics, paint flow, adhesives, accelerators, electrostatic filters, etc [1].

The Falkner-Skan equation describing this proprieties were studied from various points of view: some approximate procedures to solve a boundary layer equations [2], numerical solution [3], existence of a unique smooth solution [4], [5] and [6], was analytically investigated [7] and [8], by using Adomian decomposition method [9] and [10], etc.

The aim of the present paper is to propose a geometrical point of view and an accurate approach to Falkner-Skan equation using an analytical technique, namely optimal homotopy asymptotic method [11], [12], [13].

The validity of our procedure, which does not imply the presence of a small parameter in the equation, is based on the construction and determination of the auxiliary functions combined with a convenient way to optimally control the

[1]MSC (2010): 34-XX, 34A26, 34H05, 34M45, 35A24, 37C10, 49J15, 49K15, 65Lxx, 93C15, 93D05

convergence of the solution. The efficiency of the proposed procedure is proves while an accurate solution is explicitly analytically obtained in an iterative way after only one iteration.

From the geometry point of view, we establish the equilibrium states of the studied system and define a control function. Using specific Hamilton-Poisson geometry methods, namely the energy-Casimir method [14] we are able to study the nonlinear stability of these equilibrium states.

In this paper, a control function is proposed in order to study the stability of the equilibrium states of the system and the numerical integration via the Optimal Homotopy Asymptotic Method of the controlled system is presented.

The paper is organized as follows: in the second paragraph we put the Falkner-Skan equation in a differential system form and find the equilibrium states of the system. In the third section we find a control which preserves the equilibrium states of the system and give a Hamilton-Poisson realization of a controlled system. The fourth section is dedicated to study of stability of the controlled system. In a fifth paragraph is given a Lax formulation for the controlled system and finally in the sixth section a briefly presentation of the Optimal Homotopy Asymptotic Method, developed in [13] and used in the last part in order to obtain the approximate analytic solutions of the controlled system.

2 The Falkner-Skan equation in the flow of a viscous fluid

The dimensionless Falkner-Skan equation in the flow of a viscous fluid can be written as [2], [3], [7], [10]:

$$X'''(t) + X(t)X''(t) + \beta\Big(1 - (X'(t))^2\Big) = 0, \tag{1}$$

with the initial and boundary conditions

$$X(0) = 0, \ X'(0) = 0, \ \lim_{t\to\infty} X'(t) = 1, \tag{2}$$

where $t > 0$, β is a measure of the pressure gradient, and prime denotes derivative with respect to t.

Using the notations:

$$X(t) = x_1(t), \quad X'(t) = x_2(t), \quad X''(t) = x_3(t),$$

the nonlinear equation Eq. (1) becomes:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = -\beta\left(1 - x_2^2\right) - x_1 x_3 \end{cases}, \qquad t > 0. \tag{3}$$

The nonlinear differential system (3) has an equilibrium state $e^M = (M, 0, 0), M \in \mathbf{R}$ iff $\beta = 0$.

3 The Hamilton-Poisson realization of the system (3)

For the beginning, let us recall very briefly the definitions of general Poisson manifolds and the Hamilton-Poisson systems.

Definition: Let M be a smooth manifold and let $C^\infty(M)$ denote the set of the smooth real functions on M. A **Poisson bracket on** M is a bilinear map from $C^\infty(M) \times C^\infty(M)$ into $C^\infty(M)$, denoted as:

$$(F, G) \mapsto \{F, G\} \in C^\infty(M), F, G \in C^\infty(M)$$

which verifies the following properties:
- skew-symmetry:

$$\{F, G\} = -\{G, F\};$$

- Jacobi identity:

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0;$$

- Leibniz rule:

$$\{F, G \cdot H\} = \{F, G\} \cdot H + G \cdot \{F, H\}.$$

Proposition: Let $\{\cdot, \cdot\}$ a Poisson structure on $\mathbf{R}^n$. Then for any $f, g \in C^\infty(\mathbf{R}^n, \mathbf{R})$ the following relation holds:

$$\{f, g\} = \sum_{i,j=1}^{n} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Let the matrix given by:

$$\Pi = [\{x_i, x_j\}].$$

Proposition: Any Poisson structure $\{\cdot, \cdot\}$ on $\mathbf{R}^n$ is completely determined by the matrix Π via the relation:

$$\{f, g\} = (\bigtriangledown f)^t \Pi (\bigtriangledown g).$$

Definition: A **Hamilton-Poisson system** on $\mathbf{R}^n$ is the triple $(\mathbf{R}^n, \{\cdot,\cdot\}, H)$, where $\{\cdot,\cdot\}$ is a Poisson bracket on $\mathbf{R}^n$ and $H \in C^\infty(\mathbf{R}^n, \mathbf{R})$ is the energy (Hamiltonian). Its dynamics is described by the following differential equations system:

$$\dot{x} = \Pi \cdot \nabla H$$

where $x = (x_1, x_2, ...x_n)^t$.

Definition: Let $\{\cdot,\cdot\}$ a Poisson structure on $\mathbf{R}^n$. A **Casimir** of the configuration $(\mathbf{R}^n, \{\cdot,\cdot\})$ is a smooth function $C \in C^\infty(\mathbf{R}^n, \mathbf{R})$ which satisfy:

$$\{f, C\} = 0, \forall f \in C^\infty(\mathbf{R}^n, \mathbf{R}).$$

Let us employ the control $u \in C^\infty(\mathbf{R}^3, \mathbf{R})$,

$$u(x_1, x_2, x_3) = (0\ ,\ x_1 x_2\ ,\ -x_2^2 - x_1^2 x_2), \tag{4}$$

for the system (3). The controlled system (3)−(4), explicitly given by:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 + x_1 x_2 \qquad , \\ x_3' = -x_1 x_3 - x_2^2 - x_1^2 x_2. \end{cases} \qquad t > 0, \tag{5}$$

Proposition: The controlled system (5) has the Hamilton-Poisson realization

$$(\mathbf{R}^3, \Pi_-, H),$$

where

$$\Pi_- = \begin{bmatrix} 0 & 1 & -x_1 \\ -1 & 0 & x_2 \\ x_1 & -x_2 & 0 \end{bmatrix}$$

is the minus Lie-Poisson structure and

$$H(x_1, x_2, x_3) = \frac{1}{2} x_2^2 - x_1 x_3 - x_1^2 x_2$$

is the Hamiltonian.

Proof: Indeed, we have:

$$\Pi_- \cdot \nabla H = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

and the matrix Π_- is a Poisson matrix, see [15].

The next step is to find the Casimirs of the configuration described by the above Proposition. Since the Poisson structure is degenerate, there exist Casimir functions. The defining equations for the Casimir functions, denoted by C, are

$$\Pi^{ij}\partial_j C = 0.$$

It is easy to see that there exists only one functionally independent Casimir of our Poisson configuration, given by $C : \mathbf{R}^3 \to \mathbf{R}$,

$$C(x_1, x_2, x_3) = -x_3 - x_1 x_2.$$

Consequently, the phase curves of the dynamics Eq. (5) are the intersections of the surfaces $H(x_1, x_2, x_3) = const.$ and $C(x_1, x_2, x_3) = const.$.

4 Stability Problem

The concept of stability is an important issue for any differential equation. The nonlinear stability of the equilibrium point of a dynamical system can be studied using the tools of mechanical geometry, so this is another good reason to find a Hamilton -Poisson realization. For more details, see [15]. We start this section with a short review of the most important notions.

Definition: An equilibrium state x_e is said to be **nonlinear stable** if for each neighbourhood U of x_e in D there is a neighbourhood V of x_e in U such that trajectory $x(t)$ initially in V never leaves U.

This definition supposes well-defined dynamics and a specified topology. In terms of a norm $\| \ \|$, nonlinear stability means that for each $\varepsilon > 0$ there is $\delta > 0$ such that if

$$\|x(0) - x_e\| < \delta$$

then

$$\|x(t) - x_e\| < \varepsilon, \quad (\forall) \quad t > 0.$$

It is clear that nonlinear stability implies spectral stability; the converse is not always true.

The equilibrium states of the dynamics Eq. (1) are

$$e^M = (M, 0, 0), \quad M \in \mathbf{R}.$$

Proposition 1: For the equilibrium states $e^M = (M, 0, 0)$ the following statements hold:

a) $e^M = (M,0,0)$ are unstable for $M>0$;
b) $e^M = (M,0,0)$ are unstable for $M=0$

Proof: We will use energy-Casimir method, see [15] for details. Let

$$F_{\varphi}(x_1,x_2,x_3) = H(x_1,x_2,x_3) + \varphi[C(x_1,x_2,x_3)] =$$

$$= \frac{1}{2}x_2^2 - x_1x_3 - x_1^2x_2 + \varphi(-x_3 - x_1x_2)$$

be the energy-Casimir function, where $\varphi : \mathbf{R} \to \mathbf{R}$ is a smooth real valued function.

Now, the first variation of F_{φ} is given by

$$\delta F_{\varphi}(x_1,x_2,x_3) = x_2\delta x_2 - x_1\delta x_3 - x_3\delta x_1 - 2x_1x_2\delta x_1 - x_1^2\delta x_2 +$$

$$+\dot{\varphi}\,(-x_3 - x_1x_2)\cdot(-x_1\delta x_2 - x_2\delta x_1 - \delta x_3)$$

so we obtain

$$\delta F_{\varphi}(e^M) = [M + \dot{\varphi}\,(0)]\cdot(-M\delta x_2 - \delta x_3)$$

that is equals zero for any $M \in \mathbf{R}^*$ if and only if

$$\dot{\varphi}\,(0) = -M. \tag{6}$$

The second variation of F_{φ} at the equilibrium of interest is given by

$$\delta^2 F_{\varphi}(e^M) = [\ddot{\varphi}(0)]^{-1}\cdot[\ddot{\varphi}(0)\delta x_3 - \delta x_1 + M\cdot\ddot{\varphi}(0)\delta x_2]^2 +$$

$$+[\ddot{\varphi}(0)]^{-1}[1 + M^2\ddot{\varphi}(0) - M^2(\ddot{\varphi}(0))^2]^{-1}\cdot\Big[\Big(1 + M^2\ddot{\varphi}(0) - M^2(\ddot{\varphi}(0))^2\Big)\delta x_2 +$$

$$+\Big(M\ddot{\varphi}(0) - M\Big)\delta x_1\Big]^2 + [1+M^2\ddot{\varphi}(0) - M^2(\ddot{\varphi}(0))^2]^{-1}\cdot[-1 + M^2\ddot{\varphi}(0) - M^2]\,\Big(\delta x_1\Big)^2.$$

If we choose now φ such that the relation (6) is valid and $\delta^2 F_{\varphi}(e^M)$ is positive defined, i.e.

$$\ddot{\varphi}(0) > 0 \;\text{ and }\; 1 + M^2\ddot{\varphi}(0) - M^2(\ddot{\varphi}(0))^2 > 0 \;\text{ and }\; -1 + M^2\ddot{\varphi}(0) - M^2 > 0$$

then the second variation of F_{φ} at the equilibrium of interest is positive defined.

We can assume that $M>0$. From these inequalities we deduce that:

$$\frac{M^2+1}{M^2} < \varphi''(0) < \frac{M^2 + M\sqrt{M^2+4}}{2M^2},$$

that implies

$$2M^2+2 < M^2 + M\sqrt{M^2+4} \quad \Rightarrow \quad (M^2+2)^2 < M^2(M^2+4) \quad \Rightarrow \quad 4<0,$$

that is false.

Therefore, the equilibrium state $e^M(M,0,0)$ is unstable.

In the same way, we conclude that $e^M(M,0,0)$ is unstable for $M=0$. □

Table 1: The comparison between the approximate solutions $\bar{x}_1$ given by Eq. (7) and the corresponding numerical solutions for $\beta=0$
(relative errors: $\epsilon_{x_1} = |x1_{numerical} - \bar{x}_1|$)

t	$x1_{numerical}$	$\bar{x}_1$ given by Eq. (7)	ϵ_{x_1}
0	$1.5671 \cdot 10^{-25}$	$-1.3322 \cdot 10^{-15}$	$1.3322 \cdot 10^{-15}$
4/5	0.149674539444	0.149401535388	$2.73004 \cdot 10^{-4}$
8/5	0.582956328320	0.582978942361	$2.2614 \cdot 10^{-5}$
12/5	1.231527648000	1.231539489179	$1.1841 \cdot 10^{-5}$
16/5	1.990581010375	1.990607740256	$2.6729 \cdot 10^{-5}$
4	2.783886492275	2.783817337929	$6.9154 \cdot 10^{-5}$
24/5	3.583254092715	3.583303973631	$4.9880 \cdot 10^{-5}$
28/5	4.383220411026	4.383289581820	$6.9170 \cdot 10^{-5}$
32/5	5.183219409763	5.183234915896	$1.5506 \cdot 10^{-5}$
36/5	5.983219388168	5.983199995268	$1.9392 \cdot 10^{-5}$
8	6.783219382599	6.783194882759	$2.4499 \cdot 10^{-5}$

5 Lax formulation

Let introduce the matrices:

$$L = \begin{pmatrix} \frac{1}{2}x_2^2 & -x_1x_3 - x_3 - \frac{1}{8}x_2^4 - \frac{1}{2}(-x_1x_3 - x_1^2x_2)^2 \\ 1 & -x_1x_3 - x_1^2x_2 \end{pmatrix} ,$$

$$B = \begin{pmatrix} 1 & -x_2(x_3+x_1x_2) \\ 0 & 1 \end{pmatrix}.$$

Then an easy computation we can establish the following result:

Theorem 5.1 *The controlled system (5) have a Lax formulation, i.e., it can be put in the equivalent form:*

$$\frac{dL}{dt} = \left[L\ ,\ B\right] \quad \Leftrightarrow \quad \frac{dL}{dt} = L\cdot B - B\cdot L.$$

As in [15], the following properties hold:

$$H = Trace(L) \quad \text{and} \quad C = \frac{1}{2} Trace(L^2),$$

where H- Hamiltonian function and C- Casimir function.

Table 2: The comparison between the approximate solutions $\bar{x}_1'$ from Eq. (7) and the corresponding numerical solutions for $\beta = 0$ (relative errors: $\epsilon_{x_1'} = |x'_{1_{numerical}} - \bar{x}_1'|$)

t	$x'_{1_{numerical}}$	$\bar{x}_1'$ from Eq. (7)	$\epsilon_{x_1'}$
0	-3.8645 $\cdot 10^{-21}$	8.8817 $\cdot 10^{-16}$	8.8818 $\cdot 10^{-16}$
4/5	0.371963259413	0.372477797312	5.1453 $\cdot 10^{-4}$
8/5	0.696699514599	0.696023892471	6.7562 $\cdot 10^{-4}$
12/5	0.901065461379	0.901471382767	4.0592 $\cdot 10^{-4}$
16/5	0.980364982283	0.980092859963	2.7212 $\cdot 10^{-4}$
4	0.997770087958	0.997861518336	9.1430 $\cdot 10^{-5}$
24/5	0.999859396033	0.999974757114	1.15361 $\cdot 10^{-4}$
28/5	0.999995149208	0.999946428194	4.8721 $\cdot 10^{-5}$
32/5	0.999999902864	0.999935247532	6.4655 $\cdot 10^{-5}$
36/5	0.999999992429	0.999977995910	2.1996 $\cdot 10^{-5}$
8	0.999999993273	1.000005054164	5.0608 $\cdot 10^{-6}$

6 Numerical simulation

In this section, the accuracy and validity of the OHAM technique is proved using a comparison of our approximate solutions with numerical results obtained via the fourth-order Runge-Kutta method for $\beta = 0$.

The convergence-control parameters $K, C_i, i = \overline{1,\ 8}$ are optimally determined by means of the least-square method using the Mathematica 9.0 software.

Observation: If $\bar{x}(t)$ is the approximate analytic solution obtained via Optimal Homotopy Asymptotic Method [13], then for $\beta = 0$ the convergence-control parameters are respectively :

$$C_1 = -5.146692834756\ ,\ C_2 = -3.319352427903\ ,\ C_3 = 1.365481026558\ ,$$

$$C_4 = -0.109053890316\ ,\ C_5 = 63.014570679440\ ,\ C_6 = -183.226725640327\ ,$$

$$C_7 = 47.317886321776\ ,\ C_8 = 53.798097627583\ ,\ K = 1.679601787261\ .$$

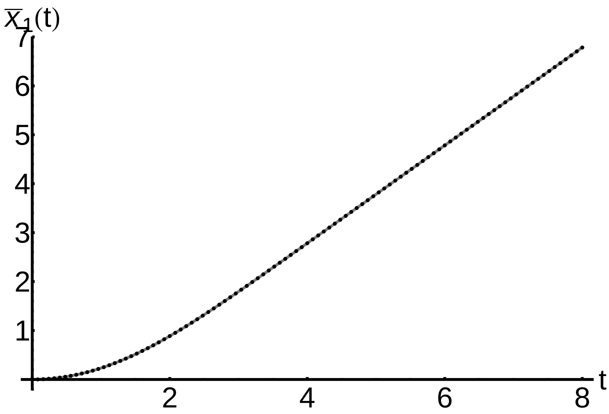

Figure 1: Comparison between the approximate solutions $\bar{x}_1$ given by Eq. (7) and the corresponding numerical solutions:
———————— numerical solution, $\cdots\cdots\cdots$ OHAM solution.

Table 3: The comparison between the approximate solutions $\bar{x}_1''$ from Eq. (7) and the corresponding numerical solutions for $\beta = 0$ (relative errors: $\epsilon_{x_1''} = |x''_{1_{numerical}} - \bar{x}_1''|$)

t	$x''_{1_{numerical}}$	$\bar{x}_1''$ from Eq. (7)	$\epsilon_{x_1''}$
0	0.469599995897	0.469599895897	$1.0000 \cdot 10^{-7}$
4/5	0.451190185801	0.460093505720	$8.9033 \cdot 10^{-3}$
8/5	0.342486827279	0.342756074406	$2.6924 \cdot 10^{-4}$
12/5	0.167560529122	0.167075663254	$4.8486 \cdot 10^{-4}$
16/5	0.046370185755	0.046300824641	$6.9361 \cdot 10^{-5}$
4	0.006874039262	0.007295190897	$4.2115 \cdot 10^{-4}$
24/5	0.000538393988	0.000306981990	$2.3141 \cdot 10^{-4}$
28/5	0.000022211398	$-9.0012 \cdot 10^{-5}$	$1.1222 \cdot 10^{-4}$
32/5	$4.7872 \cdot 10^{-7}$	$4.2932 \cdot 10^{-5}$	$4.2454 \cdot 10^{-5}$
36/5	$4.0075 \cdot 10^{-9}$	$4.9247 \cdot 10^{-5}$	$4.9243 \cdot 10^{-5}$
8	$6.8538 \cdot 10^{-10}$	$1.8697 \cdot 10^{-5}$	$1.8696 \cdot 10^{-5}$

The first-order approximate solutions proposed in [13] becomes:

$$\bar{x}_1(t) = -1.216776769791 - 0.236541146259 \cdot e^{-6.718407149045t} + t+$$
$$+e^{-5.0388805361784t} \cdot (2.067180899886 + 0.318125112302t - 0.554796671887t^2)+$$
$$+e^{-3.3592035774522t} \cdot (-4.570156113663 - 5.624329520194t - 8.333219504464t^2-$$
$$-4.761785285840t^3) + e^{-1.6796017787261t} \cdot (3.956293129828 + 4.426059140587t-$$
$$-1.432577756121t^2 - 0.407499225829t^3 + 0.162858423313t^4 - 0.012985684004t^5)\ . \tag{7}$$

Finally, Tables 1 - 3 and Figs. 1-2 emphasize the accuracy of the OHAM technique by comparing the approximate analytic solutions $\bar{x}_1$, $\bar{x}_1'$ and $\bar{x}_1''$ respectively presented above with the corresponding numerical integration values.

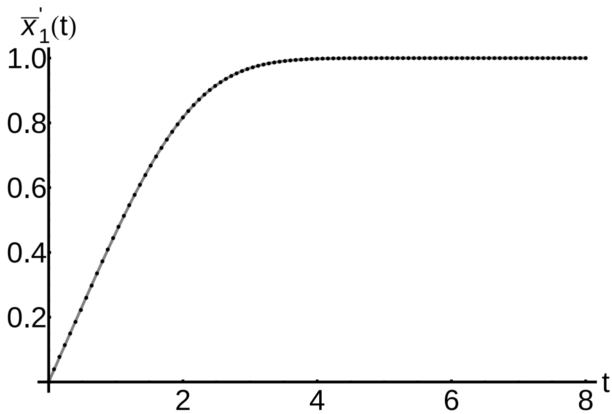

Figure 2: Comparison between the approximate solutions $\bar{x}_1'$ from Eq. (7) and the corresponding numerical solutions: ———— numerical solution, ··········· OHAM solution.

7 Conclusion

In this paper we analyze the Falkner-Skan equations from some geometrical point of view. The stability of a nonlinear differential problem governing the Falkner-Skan equation is investigated. Finding a Hamilton-Poisson realization, the results were obtained using specific tools, such as the energy-Casimir method. We give find a Lax formulation for the studied system.

Finally, the analytical integration of the nonlinear system (obtained via the Optimal Homotopy Asymptotic Method and presented in [13]) is compared with the exact solution (obtained as intersections of the surfaces $H(x_1, x_2, x_3) = const.$ and $C(x_1, x_2, x_3) = const$).

Numerical integration of the controlled dynamics is obtained via the Optimal Homotopy Asymptotic Method. Numerical simulations and a comparison with Runge-Kutta 4 steps integrator are presented, too.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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SOLVING FRACTIONAL ORDINARY DIFFERENTIAL EQUATION USING PLSM

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Abstract

In this paper, we obtaining analytical approximate solutions for fractional ordinary differential equations using *Polynomial Least Square Method* (*PLSM*). An example is illustrated to show the presented methods efficiency and convenience. [1]

Keywords and phrases: *Fractional ordinary differential equations, Polynomial Least Square Method(PLSM), Caputos fractional derivative*

1 Introduction

In recent years, fractional ordinary differential equations have been investigated by many authors. Fractional ordinary differential equations are generally used in many branches of science such as: mathematics, physics, chemistry and engineering.

Since most of these equations have no exact solutions, it has been necessary to develop numerical methods or analytical methods to find the approximate solutions of these equations.

In order to find approximate solutions of these equations, many methods were proposed, such as:

- Fractional Adams-Bashforth-Moulton method [2];
- Adomian decomposition method [4];
- Homotopy analysis method [3], [8];
- Variational iteration method [9], [10].

We consider the following fractional ordinary differential equation:

$$D^{\alpha}y(x) = f(x, y(x)) \tag{1}$$

[1]MSC (2010): 60H20, 34F15

$\alpha > 0$, with the initial condition:

$$y(0) = \nu_0 \tag{2}$$

where ν_0 are real constant and D^α denote the Caputo's fractional derivative:

$$D^\alpha \tilde{y}(x) = \frac{1}{\Gamma(n-\alpha)} \cdot \int_0^x (x-\zeta)^{n-\alpha-1} \cdot \tilde{y}^{(n)}(\zeta) d\zeta$$

$n - 1 < \alpha < n$ where $n \in \mathbb{N}^*$.

In the next section we will introduce the *Polynomial Least Square Method* (*PLSM*) which allows us to determine analytical approximate polynomial solutions for fractional ordinary differential equations and in the third section we will compare our approximate solutions with approximate solutions presented by *fractional Adams-Bashforth-Moulton method* (*FABMM*).

2 The Polynomial Least Squares Method

We denote by $\tilde{y}$ an approximate solution of equation (1). The error obtained by replacing the exact solution y with the approximation $\tilde{y}$ is given by the remainder:

$$\mathcal{R}(x, \tilde{y}(x)) = D^\alpha \tilde{y}(x) - f(x, \tilde{y}(x)). \tag{3}$$

For $\epsilon \in \mathbb{R}_+$, we will compute approximate polynomial solutions $\tilde{y}$ of the problem $(1, 2)$ on the interval $[0, b]$.

Definition 2.1. *We call an* ϵ-approximate polynomial solution *of the problem* $(1, 2)$ *an approximate polynomial solution* $\tilde{y}$ *satisfying the relations*

$$|\mathcal{R}(\tilde{y})| < \epsilon \tag{4}$$

$$\tilde{y}(0) = \nu_0. \tag{5}$$

We call a *weak ϵ-approximate polynomial solution* of the problem $(1, 2)$ an approximate polynomial solution $\tilde{y}$ satisfying the relation:

$$\int_0^b |\mathcal{R}(\tilde{y})| dx \leq \epsilon \tag{6}$$

together with the initial conditions (5).

Definition 2.2. *Let* $P_m(x) = c_0 + c_1x + c_2x^2 + \cdots + c_mx^m,\ c_i \in \mathbb{R},\ i = \overline{0, m}$ *be a sequence of polynomials satisfying the condition:*

$$P_m(0) = \nu_0.$$

We call the sequence of polynomials $P_m(x)$ *convergent to the solution of the problem* $(1, 2)$ *if* $\lim\limits_{m\to\infty} D(P_m(x)) = 0$.

We observe that from the hypothesis of the initial problems $(1, 2)$ it follows that there exists a sequence of polynomials $P_m(x)$ which converges to the solution of the problem.

We will compute a weak ϵ - approximate polynomial solution, in the sense of the Definition 2.1, of the type:

$$\tilde{y}(x) = \sum_{k=0}^{m} d_k x^k \tag{7}$$

where $d_0, d_1, \cdots, d_m$ are constants which are calculated using the following steps:

- By substituting the approximate solution (7) in the equation (1) we obtain the expression:
$$\mathcal{R}(\tilde{y}) = D^\alpha \tilde{y}(x) - f(x, \tilde{y}(x)). \tag{8}$$
If we could find $d_0, d_1, \cdots, d_m$ such $\mathcal{R}(\tilde{y}) = 0,\ \tilde{y}(0) = \nu_0$, then by substituting $d_0, d_1, \cdots, d_m$ in (7) we obtain the solutions of equation (1).

- Then we attach to the problem (1,2) the following functional:
$$\mathcal{J}(d_1, d_2, d_3, \cdots, d_m) = \int_0^b \mathcal{R}^2(\tilde{y})dx \tag{9}$$
where d_0 is computed as functions of $d_1, d_2, d_3, \cdots, d_m$ using the initial condition (5).

- We compute the values $d_1^0, d_2^0, d_3^0, \cdots, d_m^0$ as the values which give the minimum of the functional $\mathcal{J}$, and the values of d_0 is function of $d_1^0, d_2^0, d_3^0, \cdots, d_m^0$ using the initial condition.

- With constants $d_1^0, d_2^0, d_3^0, \cdots, d_m^0$ previously determined we consider the polynomial:
$$M_m(x) = \sum_{k=0}^{m} d_k^0 x^k. \tag{10}$$

Theorem 2.1. *The sequence of polynomials $M_m(x)$ from (10) satisfies the property:*

$$\lim_{x\to\infty} \int_0^b \mathcal{R}^2(M_m(x))dx = 0. \tag{11}$$

Moreover, $\forall \epsilon > 0,\ \exists m_o \in \mathbb{N},\ m > m_0$ *it follows that* $M_m(x)$ *is a* weak ϵ-approximate polynomial solution *of the problem* $(1,2)$.

Proof. Based on the way the polynomials $M_m(x)$ are computed and taking into account the relations (8)-(11), the following inequalities are satisfied:

$$0 \le \int_0^b \mathcal{R}^2(M_m(x))dx \le \int_0^b \mathcal{R}^2(P_m(x))dx,\ \forall m \in \mathbb{N},$$

where $P_m(x)$ is the sequence of polynomials introduced in Definition 2.2.

It follows that:

$$0 \le \lim_{x\to\infty} \int_0^b \mathcal{R}^2(M_m(x))dx \le \lim_{x\to\infty} \int_0^b \mathcal{R}^2(P_m(x))dx = 0.$$

We obtain:

$$\lim_{x\to\infty} \int_0^b \mathcal{R}^2(M_m(x))dx = 0.$$

From this limit we obtain that $\forall \epsilon > 0,\ \exists m_o \in \mathbb{N},\ m > m_0$ it follows that $M_m(x)$ is a *weak ϵ-approximate polynomial solution* of the problem $(1,2)$. □

In order to find ϵ-approximate polynomial solutions of the problem (1,2) by using the Polynomial Least Squares Method we will first determine weak approximate polynomial solutions, $\tilde{y}$.

If $|\mathcal{R}(\tilde{y})| < \epsilon$ then $\tilde{y}$ is also an ϵ approximate polynomial solution of the problem.

3 Application

We consider the following linear fractional differential equation ([2]):

$$D^\alpha y(x) + y(x) - x^{\alpha+3} - \frac{\Gamma(4+\alpha)}{6} \cdot x^3 = 0 \tag{12}$$

$\alpha = 0,25$; $x \in [0, \frac{1}{30}]$ and the initial condition: $y(0) = 0$.

The exact solution of the problem is:

$$y(x) = x^{3+\alpha}.$$

A numerical solutions for this problem is presented by Baskonus at all in [2] using *fractional Adams-Bashfort-Moulton method* (*FABMM*).

Using (*PLSM*):

- We compute a solution of the type:

$$\tilde{y}(x) = d_0 + d_1 \cdot x^1 + d_2 \cdot x^2 + d_3 \cdot x^3 + d_4 \cdot x^4$$

with initial condition: $\tilde{y}(0) = 0$ we obtain: $d_0 = 0$.

- The approximate solution becomes:

$$\tilde{y}(x) = d_1 \cdot x^1 + d_2 \cdot x^2 + d_3 \cdot x^3 + d_4 \cdot x^4.$$

- The corresponding remainder is:

$$\mathcal{R}(x) = \frac{4x^{3/4}\left(385d_1 + 8x\left(55d_2 + 60d_3x + 64d_4x^2\right)\right)}{1155\Gamma\left(\frac{3}{4}\right)} + \\ + d_1x + d_2x^2 + d_3x^3 + d_4x^4 - x^{13/4} - \frac{1}{6}x^3\Gamma\left(\frac{17}{4}\right). \qquad (13)$$

Next we compute:

$$\mathcal{J}(d_1, d_2, d_3, \cdots, d_m) = \int_0^{\frac{1}{30}} \mathcal{R}^2(\tilde{y})dx$$

and minimize it obtaining the values:

$$d_1 = 3,53901 \cdot 10^{-6};\ d_2 = 0,00131029;\ d_3 = 0,387136,\ d_4 = 2,29079.$$

- The approximate analytical solution of the problem (12) using (*PLSM*) is:

$$\tilde{y}(x) = 3,53901 \cdot 10^{-6} \cdot x + 0,00131029 \cdot x^2 + 0,387136 \cdot x^3 + 2,29079 \cdot x^4.$$

Table 1 present the comparison between absolute errors coresponding to the numerical solution proposed by Baskonus in [2] using (*FABMM*) and aur solution (*PLSM*).

From the table, it is easy to see that using (*PLSM*) results are better than using (*FABMM*).

Additionally, (*PLSM*) obtains the analytical solution of the polynomial form of the problem, not only numerical solutions, thus demonstrating the usefulness and accuracy of the (*PLSM*).

Table 1: Numerical results

x	$Exactsolution$	$Error(FABMM)$	$Error(PLSM)$
0.0033333	2.82×10^{-3}	3.8343×10^{-9}	2.9598×10^{-9}
0.0066667	1.73×10^{-3}	2.1194×10^{-8}	7.4355×10^{-11}
0.0100000	3.31×10^{-4}	5.4419×10^{-8}	1.8279×10^{-9}
0.0133333	1.15×10^{-3}	1.0405×10^{-7}	1.1658×10^{-9}
0.0166667	1.75×10^{-3}	1.7047×10^{-7}	6.2667×10^{-10}
0.0200000	2.36×10^{-3}	2.5705×10^{-7}	1.8004×10^{-9}
0.0233333	1.49×10^{-3}	3.5512×10^{-7}	1.2389×10^{-9}
0.0266667	2.66×10^{-3}	4.7380×10^{-7}	7.2161×10^{-10}
0.0300000	4.88×10^{-3}	6.1050×10^{-7}	1.7042×10^{-9}
0.0333333	0	7.6535×10^{-7}	3.1652×10^{-9}

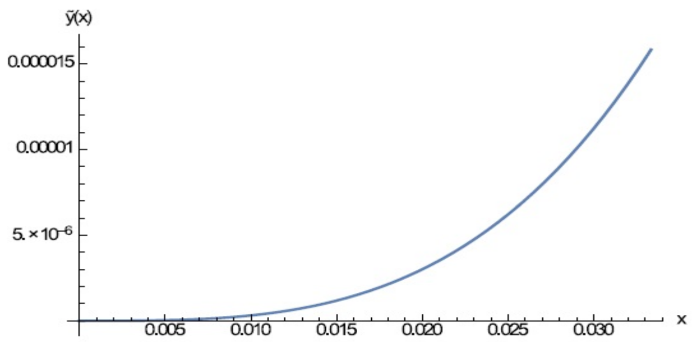

Figure 1 - The approximate analytical solution using (*PLSM*)

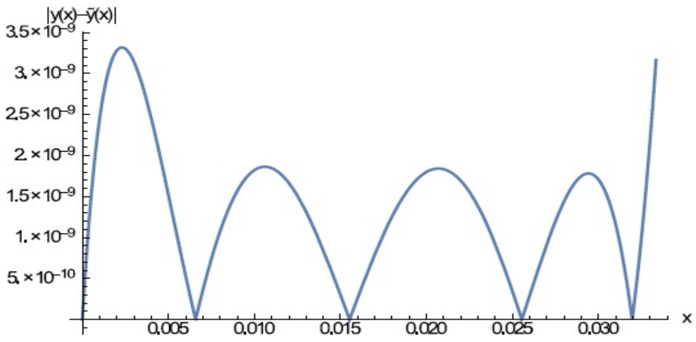

Figure 2 - The absolute errors corresponding to the approximations given by (*PLSM*)

4 Conclusions

The computations performed show that (*PLSM*) allows us to obtain approximations with an error relative to the exact or numerical solution smaller than the errors obtained using by *fractional Adams-Bashforth-Moulton method* (*FABMM*).

The application presented emphasize the high accuracy of the method by means of a comparison with previous results.

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