SCIENTIFIC BULLETIN
OF
THE POLITEHNICA UNIVERSITY OF TIMISOARA

Vol. 63(77), Issue 2 MATHEMATICS–PHISYCS 2018

SOME CONCEPTS OF FRACTIONAL DIFFERENTIAL CALCULUS USING MATLAB

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Abstract

In the last years researches in fractional calculus was extended in many areas. For further study of its applications in Machanical Area this paper presents numerical methods for solving some differential fractional equations using MATLAB. This work contains methods for fractional calculus computations like "Grünwald-Letnikov method" or "Podlubny's matrix approach" and examples using MATLAB for solving ordinary fractional differential equations. [1]

Keywords and phrases: *fractional calculus, MATLAB, differential fractional equations, Grünwald-Letnikov*

1 Introduction

In the past few years, fractional computation has become a field of study that has been searched for, in the sense of applying to different branches of science [13] such as:

- Fractals [2]
- Propagation of ultrasonic waves [8, 21]
- The theory of viscoelasticity [22]
- Fluid Mechanics [12]

The concept of fractional computation appeared in 1965 and L'Hospital wrote to Leibnitz asking him the meaning of the derivative $\frac{d^n y}{dx^n}$ if $n = \frac{1}{2}$. But if n were fractional, irrational or complex?
Leibnitz replied:

[1]MSC (2010): 34A08; 35R11; 65K15

If $n = \frac{1}{2}$ then

$$d^{\frac{1}{2}}x = x\sqrt{dx : x} \tag{1.1}$$

and "a seeming paradox from which one day will draw very useful consequences". Thus, the name of fractional computation has become an improper term for integration and arbitrary differentials.

In 1812 Laplace defined the arbitrary fractional derivatives as they were published in the writings of Lacroix's 1819.

Starting from $y = x^m,\ m \in \mathbb{Z}_+$ Lacroix has developed the following $n - th$ derivative:

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!}x^{m-n},\ m \geq n \tag{1.2}$$

Using the Legendre symbol for factorial, Gamma Function, (see Remark 1.1) will get:

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n} \tag{1.3}$$

If $y = x$ şi $n = \frac{1}{2}$ we have:

$$\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = \frac{2\sqrt{x}}{\sqrt{\pi}} \tag{1.4}$$

Remark 1.1. Definition of the Gamma function

The most important function of fractional calculation is the Function $\Gamma(z)$ as it is presented in [16]. It generalizes $n!$ and allows number n to take different values of whole numbers even complex.

Definition 1.2. *The function $\Gamma(z)$ is defined by means of the integral:*

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\ dt,$$

which converges to the right half of the Complex $Re(z) > 0$.

Indeed, we have

$$\begin{aligned}
\Gamma(x+iy) &= \int_0^\infty e^{-t}t^{x-1+iy}\ dt \\
&= \int_0^\infty e^{-t}t^{x-1}e^{iy\log(t)}\ dt \qquad (1.5)\\
&= \int_0^\infty e^{-t}t^{x-1}[\cos(y\log(t)) + i\sin(y\log(t))]\ dt.
\end{aligned}$$

The expression in square brackets is bordered fo $\forall\, t$. Convergence to infinity is given by $t = 0$ we have $x = Re(z) > 1$.

We use (1.3) to evaluate the fractional derivative of $f(t) = e^t$.

$$f(t) = e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \quad \text{(series)} \tag{1.6}$$

Applying (1.3) we obtain:

$$\frac{d^\nu}{dt^\nu} = \sum_{k=0}^{\infty} \frac{t^{k-\nu}}{\Gamma(k-\nu+1)}$$

where $\nu > 0$ and $\nu \in \mathbb{R}$ (real number) Fractional derivative of exponential function does not returns exponential function.

2 Definitions for fractional calculation

This section introduces the main definitions for fractional calculation applied in the analysis.

Definition 2.1. *Euler (1730)*

$$\frac{d^n x^m}{dx^n} = m(m-1)(m-2)...(m-n+1)x^{m-n}$$

$$\Gamma(m+1) = m(m-1)...(m-n+1)\Gamma(m-n+1)$$

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$$

$$\frac{d^{1/2} x}{dx^{1/2}} = \sqrt{\frac{4x}{\pi}} = \frac{2}{\pi} x^{1/2},$$

unde $\Gamma(z) = \int\limits_0^\infty e^{-t} t^{z-1}\, dt,\ Re(z) > 0.$

Definition 2.2. *J. B. J. Fourier (1820-1822) introduced:*

$$f(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} f(z)\, dz \int\limits_{-\infty}^{\infty} \cos(px - pz)\, dp.$$

The definition of fractional operation was obtained from the representation of the integral $f(x)$.

For n integer number, we have

$$\frac{d^n}{dx^n}\cos p(x-z) = p^n \cos[p(x-z) + \frac{1}{2}n\pi],$$

meaning

$$\frac{d^n f(x)}{dx^n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)\ dz \int_{-\infty}^{\infty} p^n \cos[p(x-z) + n\frac{\pi}{2}]\ dp.$$

Definition 2.3. *N. H. Abel (1823-1826) introduced the definition of fractional integrals:*

$$\int_0^x \frac{S'(\eta)d\eta}{(x-\eta)^\alpha} = \psi(x).$$

In fact he has solved the whole for an arbitrary number α and not just for $\alpha = \frac{1}{2}$ obtaining:

$$S(x) = \frac{\sin(\pi\alpha)}{\pi}x^\alpha \int_0^1 \frac{\psi(xt)}{(1-t)^{1-\alpha}}dt.$$

After which Abel expressed the resulting solution with the help of the α. order:

$$S(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d^{-\alpha}\psi(x)}{dx^{-\alpha}}.$$

Abel applied the fractional calculation in the solution of the integral equation of the formulation the problem of finding the shape of the curve so that the time of frictionless descent, sliding under the action of gravity independent of the point starting. If the slip time is constantly known (T), the equation becomes:

$$k = \int_0^x (x-t)^{-1/2} f(t)dt.$$

This equation, except $\frac{1}{\Gamma(1/2)}$, is the particular case of the defined integrability represents the first fraction integral $\frac{1}{2}$.

$$\sqrt{\pi}[d^{-1/2}/dx^{-1/2}]f(x)$$

$d^{1/2}/dx^{1/2}$, we get

$$\frac{d^{1/2}}{dx^{1/2}}k = \sqrt{\pi}f(x).$$

Definition 2.4. *J. Liouville (1823-1855):*

I. In its first definition, according to the exponential representation of the function $f(x) = \sum\limits_{n=0}^{\infty} c_n e^{a_n x}$, *generalized the formula* $\frac{d^m e^{ax}}{dx^n} = a^m e^{ax}$ *like*

$$\frac{d^\nu f(x)}{dx^\nu} = \sum_{n=0}^{\infty} c_n a_n^\nu e^{a_n x}$$

II. The second type of definition was that of the fractional integral:

$$\int^\mu \Phi(x) dx^\mu = \frac{1}{(-1)^\mu \Gamma(\mu)} \int_0^\infty \Phi(x+\alpha) \alpha^{\mu-1} d\alpha$$

$$\int^\mu \Phi(x) dx^\mu = \frac{1}{\Gamma(\mu)} \int_0^\infty \Phi(x-\alpha) \alpha^{\mu-1} d\alpha$$

Substituting $\tau = x + \alpha$ *şi* $\tau = x - \alpha$ *in the formulas above, obtain:*

$$\int^\mu \Phi(x) dx^\mu = \frac{1}{(-1)^\mu \Gamma(\mu)} \int_x^\infty (\tau - x)^{\mu-1} \Phi(\tau) d\tau$$

$$\int^\mu \Phi(x) dx^\mu = \frac{1}{\Gamma(\mu)} \int_{-\infty}^x (x - \tau)^{\mu-1} \Phi(\tau) d\tau.$$

III. The third definition, introduced the fractional derivative:

$$\frac{d^\mu F(x)}{dx^\mu} = \frac{(-1)^\mu}{h^\mu} \left(F(x) \frac{\mu}{1} F(x+h) + \frac{\mu(\mu-1)}{1 \cdot 2} F(x+2h) - ... \right)$$

$$\frac{d^\mu F(x)}{dx^\mu} = \frac{1}{h^\mu} \left(F(x) \frac{\mu}{1} F(x-h) + \frac{\mu(\mu-1)}{1 \cdot 2} F(x-2h) - ... \right)$$

Definition 2.5. *G. F. B. Riemann (1847-1876):*
Its definition for fractional integrals is:

$$D^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t) dt + \psi(t)$$

Definition 2.6. *N. Ya. Sonin (1869), A. V. Letnikov (1872), H. Laurent (1884), N. Nekrasove (1888), K. Nishimoto (1987):*
They considered the integral Cauchy formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_c \frac{f(t)}{(t-z)^{n+1}} dt$$

and replace n cu ν got

$$D^\nu f(z) = \frac{\Gamma(\nu+1)}{2\pi i} \int_c^{x^+} \frac{f(t)}{(t-z)^{\nu+1}} dt.$$

Definition 2.7. *Definition Riemann-Liouvill:*
The classic definition of fractional calculation is the one that shows the link between the two previous definitions.

$$_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int\limits_a^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}$$
$$(n-1 \leq \alpha < n)$$

Definition 2.8. *Grünwald-Letnikove:*
This is another definition that is sometimes useful.

$$_aD_t^\alpha f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh)$$

Definition 2.9. *M. Caputo (1967):*
The second common definition is

$$^C_aD_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)} \int\limits_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}$$
$$(n-1 \leq \alpha < n)$$

Definition 2.10. *Oldham and Spanier (1974):*

$$\frac{d^q f(\beta x)}{dx^q} = \beta^q \frac{d^q f(\beta x)}{d(\beta x)^q}$$

Definition 2.11. *K. S. Miller, B. Ross (1993):*
They used a different operator D as

$$D^{\bar{\alpha}} f(t) = D^{\alpha_1} D^{\alpha_2} ... D^{\alpha_n} f(t), \quad \bar{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$$

where D^{α_i} is definition of Riemann-Liouvill or Caputo.

3 Fractional derivatives for some special functions

1. **Unit function:** For $f(x) = 1$ we have

$$\frac{d^q 1}{dx^q} = \frac{x^{-q}}{\Gamma(1-q)}, \ \forall q.$$

2. **The identical function:** For $f(x) = x$ we have

$$\frac{d^q x}{dx^q} = \frac{x^{1-q}}{\Gamma(2-q)}.$$

3. **The exponential function:** $f(x) = e^x$ is

$$\frac{d^q e^{\pm x}}{dx^q} = \sum_{k=0}^{\infty} \frac{x^{k-q}}{\Gamma(k-q+1)}.$$

4. **The sinus function:** If $f(x) = \sin x$ then

$$\frac{d^q \sin(x)}{dx^q} = \sin\left(x + \frac{q\pi}{2}\right).$$

5. **The cosinus function:** If $f(x) = \cos x$ then

$$\frac{d^q \cos(x)}{dx^q} = \cos\left(x + \frac{q\pi}{2}\right).$$

6. Fractional derivatives $_{L}D_{+}^{\alpha}$ according to Liouville for some functions special

$f(x)$	$\frac{d^\alpha}{dx^\alpha}f(x)$
e^{kx}	$k^\alpha e^{kx} \quad k \geq 0$
$\sin(kx)$	$k^\alpha \sin(kx + \frac{\pi}{2}\alpha)$
$\cos(kx)$	$k^\alpha \cos(kx + \frac{\pi}{2}\alpha)$
$erf(kx)$	divergent
e^{-kx^2}	$\frac{k^{\frac{\alpha}{2}}}{\Gamma(1-\alpha)}(\Gamma(1-\frac{\alpha}{2})_1F_1(\frac{1}{2}+\frac{\alpha}{2};\frac{1}{2};-kx^2)$ $-\sqrt{k}\alpha x\Gamma(\frac{1}{2}-\frac{\alpha}{2})_1F_1(1+\frac{\alpha}{2};\frac{3}{2};-kx^2))$ $-2\sqrt{k}\alpha x\Gamma(\frac{3}{2}-\frac{\alpha}{2});{}_1F_1(\frac{1}{2}+\frac{\alpha}{2};\frac{3}{2};-kx^2)$ $-\frac{2}{3}k(1-\alpha^2)x^2\Gamma(\frac{1}{2}-\frac{\alpha}{2})_1F_1(\frac{3}{2}+\frac{\alpha}{2};\frac{5}{2};-kx^2))$
$_pF_q(\{a_i\};\{b_j\};kx)$	$k^\alpha \prod_{i=1}^{p}\frac{\Gamma(a_i+\alpha)}{\Gamma(a_i)}\prod_{j=1}^{q}\frac{\Gamma(b_j)}{\Gamma(b_j+\alpha)}$ $_pF_q(\{a_i+\alpha\};\{b_j+\alpha\};kx)$
$\lvert x\rvert^{-k}$	$\frac{\Gamma(k+\alpha)}{\Gamma(k)}\lvert x\rvert^{-k-\alpha}, x<0$

7. Several special functions and their fractional derivatives $_{R}D^{\alpha}$ according to the Riemann definition

$f(x)$	$\frac{d^\alpha}{dx^\alpha}f(x)$
e^{kx}	$sign(x)(sign(x)k)^\alpha$

$$e^{kx}\left(1-\frac{\Gamma(-\alpha,kx)}{\Gamma(-\alpha)}\right)$$

$\sin(kx)$	$\frac{(2-\alpha)\ ksign(x)\lvert x\rvert^{-\alpha}x}{(2-3\alpha+\alpha^2)\Gamma(1-\alpha)}\ {}_1F_2\left(1;\frac{3}{2}-\frac{\alpha}{2},2-\frac{\alpha}{2};-\frac{1}{4}k^2x^2\right)$ $-\frac{k^3sign(x)\lvert x\rvert^{-\alpha}x^3}{\left(\frac{3}{2}-\frac{\alpha}{2}\right)\left(2-\frac{\alpha}{2}\right)(2-3\alpha+\alpha^2)\Gamma(1-\alpha)}\ {}_1F_2\left(2;\frac{5}{2}-\frac{\alpha}{2},3-\frac{\alpha}{2};-\frac{1}{4}k^2x^2\right)$
$\cos(kx)$	$sign(x)\frac{\lvert x\rvert^{-\alpha}}{\Gamma(4-\alpha)}$ $((\alpha-1)(\alpha-2)(\alpha-3){}_1F_2(1;1-\frac{\alpha}{2},\frac{3}{2}-\frac{\alpha}{2};-\frac{1}{4}k^2x^2)$ $+2k^2x^2{}_1F_2(2;2-\frac{\alpha}{2},\frac{5}{2}-\frac{\alpha}{2};-\frac{1}{4}k^2x^2))$
$erf(kx)$	$-2^{-1+\alpha}k\ sign(x)\lvert x\rvert^{-\alpha}$ $((\alpha-2){}_2\bar{F}_2(\frac{1}{2},1;\frac{3}{2}-\frac{\alpha}{2},2-\frac{\alpha}{2};-k^2x^2)$ $+k^2x^2{}_2\bar{F}_2(\frac{3}{2},2;\frac{5}{2}-\frac{\alpha}{2},3-\frac{\alpha}{2};-k^2x^2))$
${}_pF_q(\{a_i\};\{b_j\};kx)$	$sign(x)\lvert x\rvert^{-\alpha}\frac{1}{\Gamma(1-\alpha)^{p+1}}F_{q+1}(\{1,1+a_i\};\{b_j,2-\alpha\};kx)$ $+ksign(x)\lvert x\rvert^{-\alpha}x\frac{1}{(1-\alpha)(2-\alpha)\Gamma(1-\alpha)}\prod_{i=1}^{p}a_i\prod_{j=1}^{q}\frac{1}{b_j}$ $\times\ {}_{p+1}F_{q+1}(\{2,1+a_i\};\{1+b_j,3-\alpha\};kx)$
$\log(x)$	$\frac{x^{-\alpha}}{\Gamma(2-\alpha)}(1-(1-\alpha)(H_{1-\alpha}+\log(x))),\ x>0$
x^k	$\frac{\Gamma(1+k)}{\Gamma(1+k-\alpha}sign(x)\lvert x\rvert^{-\alpha}x^k$

4 Method Grünwald-Letnikov

For the numerical calculation of the fractional derivatives we can use the relation:

$$_{(k-L_m/h)}D^q_{t_k}f(t)\approx h^{-q}\sum_{j=0}^{k}(-1)^j\binom{q}{j}f(t_{k-j})=h^{-q}\sum_{j=0}^{k}c_j^{(q)}f(t_{k-j}) \tag{4.1}$$

resulting from the Grünwald-Letnikov relation in Definition 2.8.

This approach is based on the fact that for most of the function classes, the definitions of Grünwald-Letnikov, Riemann-Liouvill and M. Caputo are equivalent if $f(a) = 0$.

Relationship for the explicit numerical approximation of the $q-$derivative in the points $kh, (k = 1, 2, ...)$ has the above given form (see 4.1) (Dor*čák*, 1994; Podlubny, 1999), where:
$-L_n$ is memory lenght
$-t_k = kh$
$-h$ =the time at that step
$-c_j^{(q)} (j = 0, 1, ..., k)$ = coefficient binomial.

To calculate them, we can use mathematical relations:

$$c_0^{(q)} = 1, \quad c_j^{(q)} = \left(1 - \frac{1+q}{j}\right) c_{j-1}^{(q)}. \tag{4.2}$$

Binomial coefficients $c_j^{(q)} (j = 0, 1, ..., k)$ can also be expressed factorial. By factorial writing, *Function Gamma* allows us to generalize the binomial coefficients for arguments that are not integers.

$$(-1)^j \binom{q}{j} = (-1)^j \frac{\Gamma(q+1)}{\Gamma(j+1)\Gamma(q-j+1)} = \frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}. \tag{4.3}$$

Obviously, for this simplification, the accuracy of the result is lost.

If $f(t) < M$, we can very easily set the estimated L_m (with accuracy ϵ)

$$L_m \geq \left(\frac{M}{\epsilon|\Gamma(1-q)|}\right)^{\frac{1}{q}}. \tag{4.4}$$

This is called Power Series Expansion (PSE). Transfer function discreetly resulting, the approximate fractional order operators can be expressed in the range $-z$ in the following way:

$$_0D_{kT}^{\pm r} G(z) = \frac{Y(z)}{F(z)} = \left(\frac{1}{T}\right)^{\pm r} \ PSE\left\{(1-z^{-1})^{\pm r}\right\}_n \approx T^{\mp r} R_n(z^{-1}), \tag{4.5}$$

where:

$-T-$ reference;

$-PSE\{u\}-$ The function results from the application of the PSE function u;

$-Y(z)$ is converted "Z" of the output sequence $y(kT)$;

$-F(z)$ is converted "Z" of the input sequence $f(kT)$;

$-n-$ the order of approximation;

$-R-$ the n-polynom in variable z^{-1} and $k = 1, 2, ...$

Aplication 4.1. *Let the order of fractional derivation $\alpha \in [0,1]$ for the function $y = \sin(t)$ with $t \in [0, 2\pi]$. The following code in MATLAB uses the command f deriv () entered by Bayat (2007) and based on 4.1.*

Input data:

```
clear all; close all;
h = 0.01; t = 0 : h : 2 * pi;
y = sin(t);
order = 0 : 0.1 : 1;
for i = 1 : length (order)
  yd(i, :) = fderiv(order(i), y, h);
end
[X, Y] = meshgrid (t, order);
mesh (X, Y, yd)
xlabel ('t'); ylabel ('\alpha'); zlabel ('y')
```

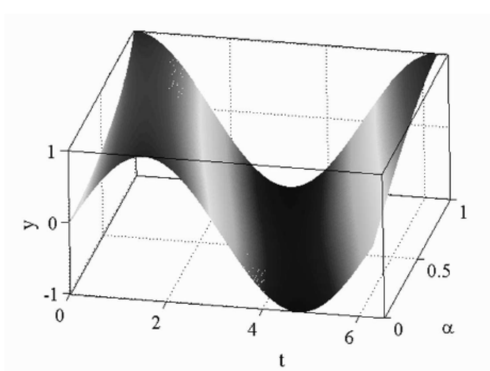

Figure 1: Fractional derivative of function $y = sin(t)$

Figure 1 is a graphical derivative of the sinus function for fractional derivation order $0 < \alpha < 1$ and $0 < t < 2\pi$.

Aplication 4.2. *We can get a better approximation of the fractional derivative if h of the first relation h is small enough so it can be demonstrated that the accuracy of this methods is* 0. *The MATLAB code for the application function 4.1 and the function* e^x.

Input:

function dy =gdiff(y, x, gam)
$h = x(2) - x(1); dy(1) = 0; y = y(:); x = x(:);$
$w = 1;$
for $j = 2 : length(x), w(j) = w(j-1)^*(1 - (gam + 1)/(j - 1));$
end
for $i = 2 : length(x), dy(i) = w(1 : i)^*[y(i : -1 : 1)]/h\hat{}gam;$
end
by Matlab code
$t = 0 : 0.001 : pi; y = sin(t); dy =$ gdiff$(y, t, 0.9); plot(t, dy)$
$t = 0 : 0.001 : pi; y = sin(t); dy =$ gdiff$(y, t, 0.9); plot(t, dy);$
hold on;
$t = 0 : 0.001 : pi; y = sin(t); dy =$ gdiff$(y, t, 0.1); plot(t, dy);$
$t = 0 : 0.001 : pi; y = sin(t); dy =$ gdiff$(y, t, 0.5); plot(t, dy);$

we get 0.1, 0.5 a 0.9 derivative of function $sin(x)$ see the fig below:

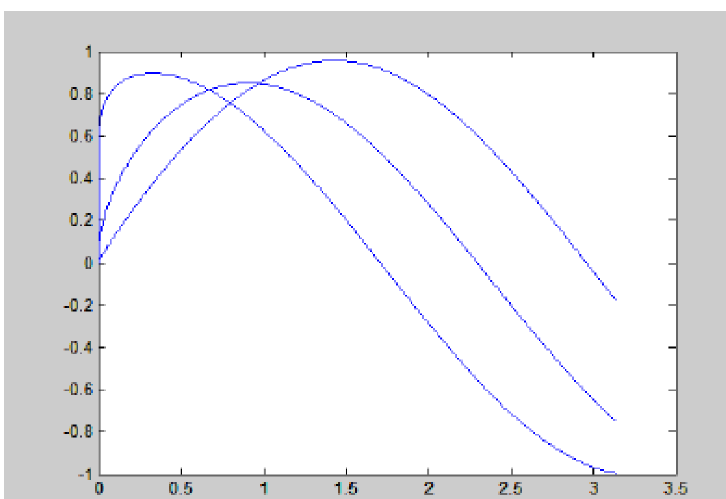

Figure 2: The $0.1, 0.5, 0.9$ derivative of function $sin(x)$

By Matlab:

$t = 0 : 0.001 : 5; y = exp(t); plot(t, y)$
$t = 0 : 0.001 : 3; y = exp(t); plot(t, y)$
hold on;
$t = 0 : 0.001 : 3; y = exp(t); dy = \text{gdiff}(y, t, 0.3); plot(t, dy)$
$t = 0 : 0.001 : 3; y = exp(t); dy = \text{gdiff}(y, t, 0.5); plot(t, dy)$
$t = 0 : 0.001 : 3; y = exp(t); dy = \text{gdiff}(y, t, 0.7); plot(t, dy)$

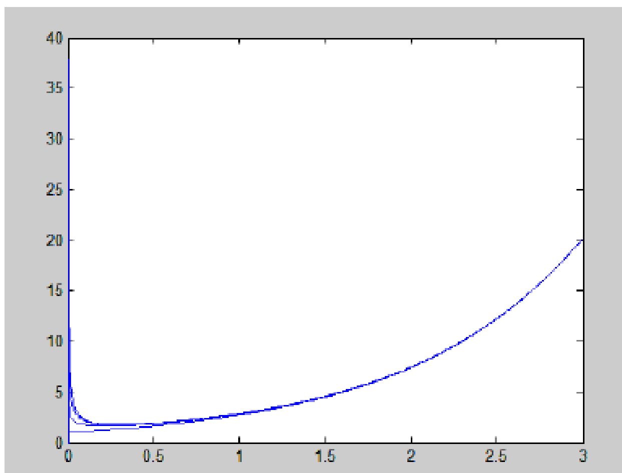

Figure 3: The $0.3, 0.5, 0.7$ derivative of function e^x

It is observed in figurative representation how $0.3-$a 0.5-a şi $0.7 - a$ derivatives of e^x are almost identical, which is similar to the classical derivation so $(e^z)' = e^z$ and e^z α derivativetimes that is also maintained for $\alpha =$fractional.

5 Differential fractional equations

The general fractional system can be described by means of the differential equation fractional form:

$$a_n D_t^{\alpha_n} y(t) + a_{n-1} D_t^{\alpha_{n-1}} y(t) + ... + a_0 D_t^{\alpha_0} y(t) = b_m D_t^{\beta_m} u(t) + b_{m-1} D_t^{\beta_{m-1}} u(t) + ... + b_0 D_t^{\beta_0} u(t), \tag{5.1}$$

where $D_t^\gamma \equiv {}_0D_t^\gamma$ express Grünwald-Letnikov, Riemann-Liouvill sau Caputo derivatives fractional. The corresponding irrational transfer function has the form

$$G(s) = \frac{b_m s^{\beta_m} + ... + b_1 s^{\beta_1} + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + ... + a_1 s^{\alpha_1} + a_0 s^{\alpha_0}} = \frac{Q(s^{\beta_k})}{P(s^{\alpha_k})}, \tag{5.2}$$

where $a_k (k = 0, ...n)$, $b_k (k = 0, ...m)$ are constants, and $\alpha_k (k = 0, ...n)$, $\beta_k (k = 0, ...m)$ are real or rational numbers of any kind and without limitation the generality may be arranged

$$\alpha_n > \alpha_{n-1} > ... > \alpha_0, \ \beta_m > \beta_{m-1} > ... > \beta_0. \tag{5.3}$$

In a particular case for systems of commensurable order, keep $\alpha_k = \alpha k$, $\beta_k = \alpha k$, $(0 < \alpha < 1)$, $\forall k \in \mathbb{Z}$, and the transfer function has the following form:

$$G(s) = K_0 \frac{\sum\limits_{k=0}^{M} b_k (s^\alpha)^k}{\sum\limits_{k=0}^{N} a_k (s^\alpha)^k} = K_0 \frac{Q(s^\alpha)}{P(s^\alpha)}. \tag{5.4}$$

With $N > M$, the function $G(s)$ becomes its own rational function in complex variables s^α and what can be extended to form:

$$G(s) = K_0 \left[\sum_{i=1}^{N} \frac{A_i}{s^\alpha + \lambda_i} \right],$$

where $\lambda_i (i = 1, 2, ..., N)$ are the roots of the pseudo polynomial or the polynomial system. The analytical solution of the system can be expressed

$$y(t) = \mathcal{L}^{-1} \left\{ K_0 \left[\sum_{i=1}^{N} \frac{A_i}{s^\alpha + \lambda_i} \right] \right\} = K_0 \sum_{i=1}^{N} A_i t^\alpha E_{\alpha,\alpha}(-\lambda_i t^\alpha), \tag{5.5}$$

$$a_n D_t^{\alpha_n} y(t) + ... + a_1 D_t^{\alpha_1} y(t) + ... + a_0 D_t^{\alpha_0} y(t) = 0, \tag{5.6}$$

where $a_k (k = 0, 1, ..., n)$ are constant coefficients; $\alpha_k (k = 0, 1, 2, ..., n)$ are real numbers.

Without restricting generality, we ca assume that $\alpha_n > \alpha_{n-1} > ... > \alpha_0 \geq 0$.

The analytical solution of 5.6 is given by the general formula in the form:

$$y(t) = \frac{1}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{k_0 + k_1 + ... + k_{n-2} = m \\ k_0 \geq, ..., k_{n-2} \geq 0}} (m; k_0, k_1, ..., k_{n-2})$$

$$\times \prod_{i=0}^{n=2} \left(\frac{a_i}{a_n} \right)^{k_i} \mathcal{E}_m(t, -\frac{a_{n-1}}{a_n}; \alpha_n - \alpha_{n-1}, \alpha_n + \sum_{j=0}^{n=2} (\alpha_{n-1} - \alpha_j) k_j + 1),$$

where $(m, k_0, k_1, ..., k_{n-2})$ are multinomial coefficients.

It is the control function that modifies the system 5.6 in:

$$a_n D_t^{\alpha_n} y(t) + ... + a_1 D_t^{\alpha_1} y(t) + a_0 D_t^{\alpha_0} y(t) = u(t). \tag{5.7}$$

Through Laplace, we get the fractional transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{a_n s^{\alpha_n} + ... + a_1 s^{\alpha_1} + a_0 s^{\alpha_0}}. \tag{5.8}$$

Aplication 5.1. *Given a fractional differential of the second order, with initial zero conditions,* $\alpha = 1.5, a = 2, b = 1, pasul = 0.001$ *calculation time* $20sec.$ *:*

$$a D_t^{\alpha} y(t) + b y(t) = 1. \tag{5.9}$$

The solution can be obtained using Laplace's transformation method, it can be expressed:

$$Y(s) = \frac{1/a}{s(s^\alpha + b/a)} \tag{5.10}$$

and the general solution is as follows:

$$y(t) = \frac{1}{a}\mathcal{E}_0(t, -\frac{b}{a}; \alpha, \alpha + 1) \equiv \frac{1}{a} t^\alpha E_{\alpha,\alpha+1}\left(-\frac{b}{a}t^\alpha\right). \tag{5.11}$$

To get the solution in MATLAB, we can use the following commands:

```
clear all; close all;
a = 2; b = 1; alpha = 1.5;
t = 0 : 0.001 : 20;
y = (1/a) * t.^(alpha). * mlf(alpha, alpha + 1, ((-b/a) * t.^(alpha)));
plot(t, y);
xlabel('Time[sec]');
ylabel('y(t)');
```

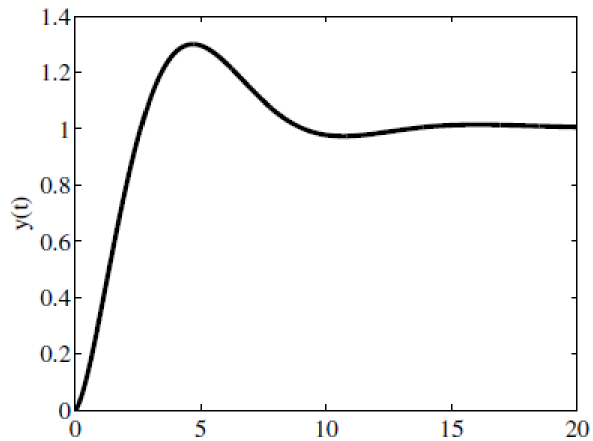

Figure 4: The equation solution

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