SCIENTIFIC BULLETIN
OF
THE POLITEHNICA UNIVERSITY OF TIMIŞOARA

Vol 61 (75), Issue 2 MATHEMATICS –PHYSICS 2016

THE CONNECTION BETWEEN KOSTANT'S GEOMETRIC QUANTIZATION AND THE SYMMETRICAL OPTIMUM METHOD FOR CONTROLLER TUNING VIA A PARTICULAR HAMILTONIAN MECHANICAL SYSTEM

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Abstract

In this paper we present a new example of Hamiltonian mechanical system which is equivalent at classic level with the two systems presented in [9], but not in quantum level with the two ones, and the way one can design a linear controller for this mechanical system using the symmetrical optimum method.[1]

1 Introduction

In this paper we will introduce a new example of Hamiltonian mechanical systems which is equivalent at classic level with the ones presented in [9], but not in quantum level.

The novelty will be given by the fact that throw this new example we can make a new connection between Kostant's geometric quantization and the automatics control. More specifically for this particular case of mechanical system we will design a Proportional-Integrator controller based on the Symmetrical Optimum Method [5].

[1]Mathematical Subject Classification (2010): *53D50, 81S10*

Keywords and phrases: *quantization, Hamiltonian mechanical system, symmetrical optimum method, linear controller.*

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2 Kostant's geometric quantization

Let M be a differential variety and (M,ω) a symplectic manifold.

Definition 2.1. ([6]) We will say that the symplectic manifold (M,ω) is quantizable if there are

(i) a complex line bundle $L^{\omega} := (L,p,M)$ over M;

(ii) a Hermitian structure $(\cdot,\cdot)$ on L^{ω};

(iii) a connection ∇^{ω} on L^{ω} compatible with the Hermitian structure

$$\text{such that } [\nabla^{\omega}_X,\nabla^{\omega}_Y] = \nabla^{\omega}_{[X,Y]} + \frac{1}{i\hbar}\omega\left(X,Y\right), \ \forall X,Y \in \mathfrak{X}_{\mathbb{C}}\left(M\right).$$

Definition 2.2. ([6]) The complex line bundle L^{ω} is called quantum bundle.

Proposition 2.1. ([6]) If (M,ω) is a quantizable manifold, then the pair $(\mathcal{H}^{\omega},\delta^{\omega})$ defines a prequantization of (M,ω).

Definition 2.3. ([6]) A distribution D on M is an application which associate at each point $x \in M$ a linear subspace $D_x \subset T_xM$ such that the next conditions can be verified:

(i) $k = \dim D_x, \ \forall x \in M$;

(ii) $\forall x_0 \in M, \ \exists \ V_{x_0} \in \mathcal{V}_{(x_0)}, \ \exists \ X_1,...,X_n \in \mathfrak{X}\left(\mathcal{V}_{x_0}\right)$ linearly independent such that $\text{Span}\left(\left(X_1\right)_x,...,\left(X_n\right)_x\right) = D_x, \ \forall x \in V_{x_0}$.

Definition 2.4. ([6]) The distribution D is integrable if for any $x \in M$, there is a submanifold N of M such that:

(i) $x \in N$;

(ii) $\dim N = k$;

(iii) $T_yN = D_y, \ \forall y \in N$.

Definition 2.5. ([6]) An integral distribution is called foliation. The maximal submanifolds $N \subset M$ for which $T_xN = D_x$ are called the foliation leaves.

Let M/D be the space of the foliation leaves D. If there is a differentiable structure M/D such that the canonical projection $\pi : M \to M/D$, be a surjective submersion, then D is called reducible foliation.

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Definition 2.6. ([6]) Let (M, ω) be a symplectic manifold of dimension $2n$. A real polarization on M is a foliation D on M such that D is maximal izotropic in relation with ω, i.e. the next conditions are satisfied:

(i) $\omega_x (D_x, D_x) = 0,\ \forall x \in M$;

(ii) no ather subspace of $T_x M$ which contains D_x has the property (i), or in other words D_x is maximal with the property (i).

In particular, $\dim D_x = n,\ \forall x \in M$. The polarization D is called reducible if the foliation D is reducible.

Example 2.1. Let $(M, \omega) = (T^* Q, \omega = d\theta)$. Then it can be easily checked that D^v and D^h are real polarizations, reductibles on M.

Proposition 2.2. ([6]) The pair $(\mathcal{H}_D^\omega, \delta_D^\omega)$ defines a prequantization of the quantizable manifold (M, ω), ie the algebra $(C^\infty (M, D; \mathbb{R}), \{\cdot, \cdot\}_\omega)$ satisfies the conditions of the Dirac's problem ([2]).

3 Ambiguities in Kostant's geometric quantization

Starting from Puta and Hedrea's idea [9], we will show that there are mechanical Hamiltonian systems which are equivalent at classic level, i.e. their Hamiltonian fields have the same integral curves (with a fixed initial condition), but they aren't equivalent at quantum level, i.e. the quantum operators corresponding to the energy have different spectra. For the systems (3.1) and (3.2) given as follows the things are made in detail in [9].

Let's consider the Hamiltonian mechanical systems

$$\left(T^*\mathbb{R} \simeq \mathbb{R}^2,\ \omega = dp \wedge dq,\ H(p,q) = q\right), \tag{3.1}$$

$$\left(T^*\mathbb{R} \simeq \mathbb{R}^2,\ \omega' = \frac{e^q}{(e^q+1)^2} dp \wedge dq,\ H'(p,q) = 1 - \frac{1}{e^q+1}\right), \tag{3.2}$$

$$\left(T^*\mathbb{R} \simeq \mathbb{R}^2,\ \omega'' = \frac{e^q - 2}{e^{2q}} dp \wedge dq,\ H''(p,q) = \frac{1 - e^q}{e^{2q}}\right). \tag{3.3}$$

Proposition 3.1. The Hamiltonian mechanical systems (3.1), (3.2) and (3.3), are equivalent at classic level.

Proof.
From the three mechanical system from the above immediately results that

$$X_H = -\frac{\partial}{\partial p},\ X_{H'} = -\frac{\partial}{\partial p},\ X_{H''} = -\frac{\partial}{\partial p},$$

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i.e.

$$X_H = X_{H'} = X_{H''},$$

which means that the there Hamiltonian mechanical systems are equivalent at classic level.

□

Proposition 3.2. The manifolds $(T^*\mathbb{R}, \omega)$, $(T^*\mathbb{R}, \omega')$, $(T^*\mathbb{R}, \omega'')$ are quantizable.

Proof.
Because we have

$$\omega = d(pdq), \ \omega' = d\left(\frac{e^q}{(e^q+1)^2}\right) pdq, \ \omega'' = d\left(\frac{e^q - 2}{e^{2q}}\right) pdq,$$

results that $(T^*\mathbb{R}, \omega)$, $(T^*\mathbb{R}, \omega')$, $(T^*\mathbb{R}, \omega'')$ are quantizable.

□

From the above proposition results that:

Proposition 3.3. For the system (3.1) we obtain ([9])

$$\begin{aligned} &L^\omega = (T^*\mathbb{R} \times \mathbb{C}, pr_1, T^*\mathbb{R}), \\ &\Gamma(L^\omega) = C^\infty(T^*\mathbb{R}, \mathbb{C}), \\ &\nabla_X^\omega f = X(f) - \frac{i}{\overline{h}}(pdq)(X) f, \\ &((x, z_1), (x, z_2)) = z_1 \overline{z}_2. \end{aligned}$$

Proposition 3.4. For the system (3.2) we obtain ([9])

$$\begin{aligned} &L^{\omega'} = (T^*\mathbb{R} \times \mathbb{C}, pr_1, T^*\mathbb{R}), \\ &\Gamma\left(L^{\omega'}\right) = C^\infty(T^*\mathbb{R}, \mathbb{C}), \\ &\nabla_X^{\omega'} f = X(f) - \frac{i}{\overline{h}}\left[\frac{e^q}{(e^q+1)^2} pdq\right](X) f, \\ &((x, z_1), (x, z_2))_x = z_1 \overline{z}_2. \end{aligned}$$

Proposition 3.5. For the system (3.3) we obtain

$$\begin{aligned} &L^{\omega''} = (T^*\mathbb{R} \times \mathbb{C}, pr_1, T^*\mathbb{R}), \\ &\Gamma\left(L^{\omega''}\right) = C^\infty(T^*\mathbb{R}, \mathbb{C}), \\ &\nabla_X^{\omega''} f = X(f) - \frac{i}{\overline{h}}\left[\frac{e^q - 2}{e^{2q}} pdq\right](X) f, \\ &((x, z_1), (x, z_2))_x = z_1 \overline{z}_2. \end{aligned}$$

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□

Theorem 3.1: If we consider on $T^*\mathbb{R}$ the vertical polarization D^v, then it is obviously reducible so we can reach the conclusion that H, H' şi H'' are quantized, but we also observe that

$$(\delta^{\omega}_{D^v})_H = q,\ \left(\delta^{\omega'}_{D^v}\right)_H = 1 - \frac{1}{e^q+1},\ \left(\delta^{\omega''}_{D^v}\right)_H = \frac{1-e^q}{e^{2q}}.$$

Then it results that

$$Spec\ (\delta^{\omega}_{D^v})_H = \mathbb{R},\ Spec\ \left(\delta^{\omega'}_{D^v}\right)_H = (0,1),\ Spec\ \left(\delta^{\omega''}_{D^v}\right)_H = \left[-\frac{1}{4},\infty\right).$$

Proof.

If the first spectrum is obvious, for the other two we make the justification and for that we consider the functions

$$f_{\omega'} : \mathbb{R} \to \mathbb{R},\ f_{\omega'}(x) = 1 - \frac{1}{e^x+1},$$

and

$$f_{\omega''} : \mathbb{R} \to \mathbb{R},\ f_{\omega''}(x) = \frac{1-e^x}{e^{2x}}.$$

For the first function it results that

$$\lim_{x\to-\infty} f_{\omega'}(x) = 0,\ \lim_{x\to\infty} f_{\omega'}(x) = 1,$$

with

$$y = 0 \text{ and } y = 1 \text{ horizontal asymptotes},$$

and

$$\begin{aligned} f'_{\omega'}(x) &= -\frac{-1}{(e^x+1)^2}\cdot e^x \\ &= \frac{e^x}{(e^x+1)^2} > 0,\ x \in \mathbb{R}. \end{aligned}$$

from where we have the following table:

Table 3.1.

	$-\infty$							$+\infty$
$f'_{\omega'}$	+	+	+	+	+	+	+	+
$f_{\omega'}$	0	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$			1

,

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For the second function we have

$$\lim_{x\to-\infty} f_{\omega''}(x) = +\infty, \quad \lim_{x\to\infty} f_{\omega''}(x) = 0,$$

with

$$y = 0 \text{ horizontal asymptote to } +\infty$$

and

$$f'_{\omega''}(x) = \frac{e^x - 2}{e^{2x}},$$

which has a local minimum point because $f'_{\omega''}(\ln 2) = 0$ and more specific $f_{\omega''}(\ln 2) = -\frac{1}{4}$, from where we obtain the following table:

Table 3.2.

	$-\infty$				$\ln 2$				$+\infty$
$f'_{\omega''}$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$
$f_{\omega''}$	$+\infty$	$\searrow$	$\searrow$		$-\frac{1}{4}$	$\nearrow$	$\nearrow$		0

□

Remark 3.1. Therefore we observe that the all the three mechanical systems are not equivalent at quantum level.

□

4 Some important definitions in Automatic Control

Definition 4.1. ([8]) An automatic control system is a closed-loop control system that can partially or totally remove the intervention of a human operator. It implies the existence of two interconnected subsystems: the control system and the controlled process.

The scheme of a closed loop control system with negative feedback is presented in Figure 4.1.

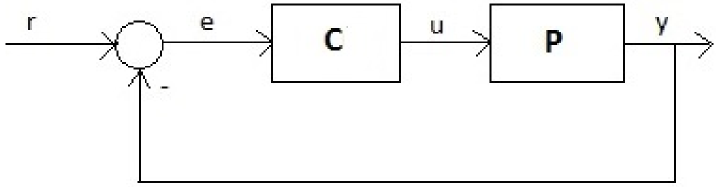

Fig. 4.1. The closed loop control system scheme.

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where C - controller, P- process, r - reference signal, e - error, u - input signal, y - output signal.

Definition 4.2. ([8]) A controller is a device that takes in the operational space the error e (obtained as the difference between the input signal r and the output signal y) and gives as its output the input signal for the process P.

Definition 4.3. ([3]) A transfer function is a mathematical representation which describes a relation between the input and the output of a system. For continuous time system, the transfer function is a linear mapping of the Laplace transform of the input to the Laplace transform of the output.

5 The Symmetrical Optimum Method (SO-m) utilized in a case study

The symmetric optimum method was introduced by C. Kessler [5] and it is used in many variants for controller tuning [[10], [4], [1]]. An extended version of the SO-m method is presented in [7]. The basic idea consists in making into the transfer function of the open loop system a second order pole in the origin which ensures null tracking error in relation to the ramp entry variations.

A version of the method is applied in the situations when the process has a integral component [10] $H_p(s) = \frac{k_p}{s(1+sT_\Sigma)(1+sT_1)(1+sT_2)}$.

Thanks to the double pole created in the origin the transfer function of the closed system has the following form [10]:

$$H_r(s) = \frac{k_r k_p T_r s + k_r k_p}{s^3 T_\Sigma + s^2 + k_r k_p T_r s + k_r k_p} = \frac{b_1 s + b_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0} \tag{5.1}$$

with

$$b_0 = a_0, b_1 = a_1, a_0 = k_r k_p, a_1 = k_r k_p T_r, a_2 = 1, a_3 = T_\Sigma.$$

In the pulse- module characteristic [10]:

$$|H_r(j\omega)| = \sqrt{\frac{a_0^2 + a_1^2\omega^2}{a_0^2 - (2a_0a_2 - a_1^2)\omega^2 - (2a_1a_3 - a_2^2)\omega^4 + a_3^2\omega^6}} \tag{5.2}$$

The setup of (5.2) allows suboptimal generalization of the method and the expression of the controller parameters can be determined. To a custom process a certain type of controller is attached [5], [10].

Case 1, for a process with the transfer function $H_P = \dfrac{k_p}{s(1+sT_\Sigma)}$ a PI controller with the transfer function $H_r(s)$ can be designed:

$$H_r(s) = \frac{k_r}{s}(1+sT_r), k_r = \frac{k_R}{T_i} = \frac{1}{8k_pT_\Sigma^2}, T_r = 4T_\Sigma \tag{5.3}$$

Case 2, for a process with the transfer function $H_P = \dfrac{k_p}{s(1+sT_\Sigma)(1+sT_1)}$ a PID controller with the transfer function $H_r(s)$ can be designed:

$$H_r(s) = \frac{k_r}{s}(1+sT_r)(1+sT_r'), k_r = \frac{1}{8k_pT_\Sigma^2}, T_r = 4T_\Sigma, T_r' = T_1, \tag{5.4}$$

Case 3, for a process with the transfer function $H_P = \dfrac{k_p}{s(1+sT_\Sigma)(1+sT_1)(1+sT_2)}$ a PID2-T1 controller with the transfer function $H_r(s)$ can be designed:

$$H_r(s) = \frac{k_r}{s}(1+sT_r)\frac{(1+sT_r')}{(1+sT_f')}\frac{(1+sT_d)}{(1+sT_f)}, k_r = \frac{1}{8k_pT_\Sigma^2}, T_r = 4T_\Sigma, \tag{5.5}$$

$$T_r' = T_1, T_d = T_2, T_d/T_f \approx 10.$$

In the following we want to show that for the third Hamiltonian mechanical system one can design a linear controller using the symmetrical optimum method.

Proposition 5.1. The Laplace transform for the third Hamiltonian mechanical system is

$$\mathcal{L}\left[\frac{1-e^q}{e^{2q}}\right] = \frac{1}{s+2} - \frac{1}{s+1} = -\frac{1}{(s+1)(s+2)}$$

This process fits in *Case 1*, so the controller designed on the symmetrical optimum method is of type Proportional-Integrator (PI) and has the transfer function $H_r(s) = \dfrac{k_r}{s}(1+sT_r)$ where $k_r = -0,12$ and $T_r = 4$.

After that we made a closed loop experiment in MATLAB/Simulink. A unitary step was applied as input signal at *t=1s*. The system output is presented in Figure 5.1. The simulation results show that the process is stable in closed loop, which means that the designed PI controller is a good one.

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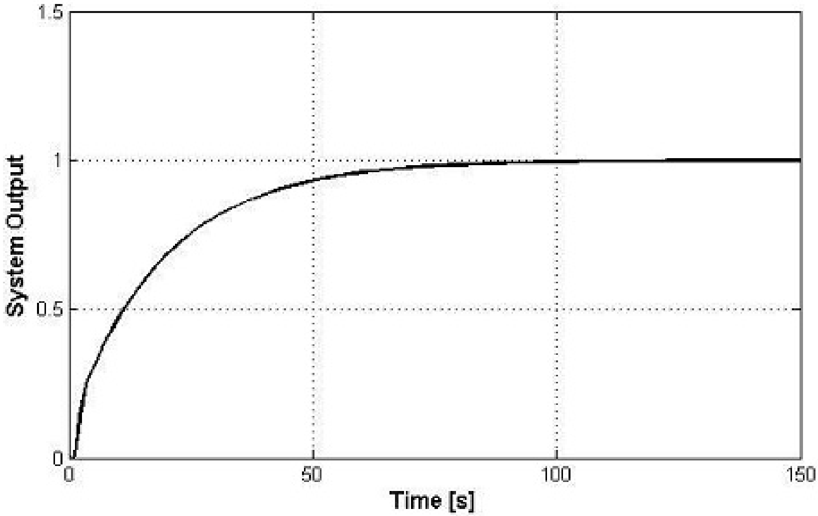

Fig. 5.1. The output of the closed loop system.

From the above chapters it results that:

Theorem 5.1. The three Hamiltonian mechanical systems are equivalent at classic level, but not at quantum level. Moreover it can be shown that for the last one a linear controller can be designed based on Kessler's symmetrical optimum method. □

6 Conclusions

Alongside the first two systems which were described in [9] a third one was build. This new system is equivalent at classic level with the first two, but all of them are different at quantum level.

The most interesting part of the article is that we have shown that there are Hamiltonian mechanical systems (whose manifolds are quantified) which can be controlled by designing a controller based on the symmetrical optimum method.

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