A Direct Scheme for Order Reduction of Linear Discrete Systems and Stabilization using Lower Order Model

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Abstract: *In this paper, a new simple and direct stability preserving scheme for the order reduction of a stable Linear Time Invariant Discrete Systems (LTIDS) is presented. The reduced order models are derived from the proposed modified Jury's table. The presented order reduction scheme is direct and the reduced model retains the stability and steady state value of the original system. To judge the effectiveness of the reduced model, a PID controller for the reduced order model is designed, tuned to meet out the desired performance specifications and then it is connected to the original higher order system. The original system with the designed PID controller shows promising results. The validity of the proposed method is illustrated through a numerical example.*

Keywords: *Linear Time Invariant Discrete System, Order reduction, Stability Equation, Modified Jury's table, PID controller.*

1. Introduction

The mathematical modeling is indispensable in analysis, design and synthesis of real time control systems. Such modeling in time domain often results in higher order differential equations which further leads to higher order transfer functions in frequency domain and corresponding higher order controllers. The cost and complexity of the controller increase with order of the system. The calculations involved in the design of controller and compensator for higher order stable LTIDS can be minimized by reducing it to a suitable lower order system. Hence it is essential to approximate a higher order system to a reduced order system in either time domain or frequency domain which must retain and reflect the important characteristics of the original system as closely as possible. Stability and steady state values are the most essential properties to be preserved by all the reduced order models.

Many methods are available for formulating reduced order models for LTIDS which include both direct and indirect techniques. Generally, these methods involve with either of bilinear transformations, Routh Pade approximants, continued fraction scheme, moment matching technique or using eigen spectrum analysis. In case of indirect method, the continuous system reduction techniques are extended to discrete time system which needs domain transformations.

Few direct order reduction (without any domain transformations) of discrete time systems had been developed. C.P.Therapos proposed a direct simple method of obtaining reduced model [1]. R.Prasad developed an order reduction technique using stability equation and weighted time moments [2]. Few other methods are briefed in [3]. A mixed method is formulated using stability equation and reciprocal transformation [4]. Numerous computer oriented reduction techniques based on Genetic algorithm, Particle Swarm Algorithm, modified Particle Swarm Algorithm and Differential Evolution Algorithm [5], [6] are projected in the recent years. However, in this paper a new, simple version of customary direct scheme is suggested to formulate the reduced order model with help of modified Jury's table. This scheme is found to be simple in application compared with other methods.

2. DESCRIPTION OF THE PROPOSED METHOD

Consider the nth order stable original higher order LTIDS represented by the transfer function G(z) as

$$\boldsymbol{G(z)} = \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{n-1} a_i z^i}{\sum_{i=0}^{n} b_i z^i} = \frac{a_{n-1}z^{n-1}+a_{n-2}z^{n-2}+\cdots\ldots+a_2z^2+a_1z+a_0}{b_nz^n+b_{n-1}z^{n-1}+\cdots\ldots+b_2z^2+b_1z+b_0} \tag{1}$$

where, N(z) is the numerator polynomial, and D(z) is the denominator polynomial of the higher order system. a_i $(0 \le i \le n-1)$ and $b_i$$(0 \le i \le n)$ are scalar constants.
The corresponding stable kth (where k<n) order reduced model is of the form

$$R_k(z) = \frac{N_k(z)}{D_k(z)} = \frac{\sum_{i=0}^{k-1} c_i z^i}{\sum_{i=0}^{k} d_i z^i} = \frac{c_{k-1}z^{k-1} + c_{k-2}z^{k-2} + \cdots \dots \dots + c_2 z^2 + c_1 z + c_0}{d_k z^k + d_{k-1}z^{k-1} + \cdots \dots \dots + d_2 z^2 + d_1 z + d_0} \quad (2)$$

Where $N_k(z)$ and $D_k(z)$ are numerator and denominator polynomials of the reduced order model respectively and c_i $(0 \le i \le k-1)$ and di $(0 \le i \le k)$ are scalar constants.

In this paper, assuming the original system described by (1), it is aimed to find a reduced order model in the form (2) such that the reduced model retains the important characteristics of the original system and approximates its response as closely as possible for same type of inputs with minimum integral square error.

The derivation of successful reduced order model coefficients for the original higher order model is done from the modified Jury table proposed in [7]. From which, the stability and the root distributions are examined [7]. Two tables are formed one is for the numerator polynomial of G(z) and other for denominator polynomial of G(z) as shown below. From each table, appropriate reduced models are extracted.

TABLE I. MODIFIED JURY TABLE FOR N(Z)

Row	Z0	Z1	Z2	…	Zn-3	Zn-2	Zn-1
1	a0	a1	a2	…	an-3	an-2	an-1
2	an-1	an-2	an-3	…	a2	a1	a0
3	e0	e1	e2	…	en-3	en-2	
4	en-2	en-3	en-4	…	e1	e0	
5	f0	f1	f2	…	fn-3		
6	fn-3	fn-4	fn-5	…	f0		
	: :	: :	: :				
2n-1	y0	y1					
2n	y1	y0					
2n+1	l0						

In the above table, the first-row elements are formed using the coefficients of the numerator polynomial N(z). Further the odd row elements are obtained as the determinants of the matrixes as shown below.

$$e_k = \begin{vmatrix} a_0 & a_{k+1} \\ a_{n-1} & a_{n-2-k} \end{vmatrix}; \quad f_k = \begin{vmatrix} e_0 & e_{k+1} \\ e_{n-2} & e_{n-3-k} \end{vmatrix}; \quad \dots\dots\dots\dots\dots\dots\dots\dots ; \quad l_0 = \begin{vmatrix} y_0 & y_1 \\ y_1 & y_0 \end{vmatrix};$$

The elements of even row (2k+2) consists of the elements of the odd row (2k+1) but in reverse order. Where k=0,1,2,3…… The process of calculating each row elements is extended till the single element row (l_0) as shown in the table I.

The reduced order polynomials can be formed from the table with the help of elements available in the odd rows. Since N(z) is a stable polynomial then the reduced order polynomials are also stable polynomials. The stability of the reduced order polynomial can be checked by examining the algebraic signs of the last elements of odd rows i.e. (e_{n-2}, f_{n-3}, ….. y_1 and l_0) in table I. Let $\delta_1=e_{n-2}$, $\delta_2=f_{n-3}$,…….. $\delta n=l_0$. The sufficient conditions for stability is that all δi values should have negative sign [7].

In the similar fashion, another table can be constructed for D(z) in terms of its coefficients as shown in table II.

TABLE II. MODIFIED JURY TABLE FOR D(Z)

Row	Z0	Z1	Z2	…	Zn-2	Zn-1	Zn
1	b0	b1	b2	….	bn-2	bn-1	bn
2	bn	bn-1	bn-2	…	b2	b1	b0
3	g0	g1	g2	…	gn-2	gn-1	
4	gn-1	gn-2	gn-3	…	g1	g0	
5	h0	h1	h2	…	hn-2		
6	hn-2	hn-3	hn-4	…	h0		
	: :	: :	: :				
2n-3	x0	x1	x2				
2n-2	x2	x1	x0				
2n-1	m0	m1					
2n	m1	m0					
2n+1	n_0						

In the above table, the elements of first and second rows are formed with the coefficient of denominator polynomial. The remaining elements are evaluated as done in the formation of the numerator table1. Using the elements of the odd rows, suitable reduced order polynomials are formulated for further analysis.

Formulation of Second Order Model

The numerator coefficients are chosen from the (2n-1)th row of table 1 and denominator coefficients are chosen from the (2n-3)th row of table II. For instance, a second order reduced model is formed from the coefficients of table I and table II as given in (3).

$$\boldsymbol{R(z)} = \frac{y_1 z + y_0}{x_2 z^2 + x_1 z + x_0} \tag{3}$$

By using the modified Jury table, the reduced order system is obtained as follows.

Step(i): Form the modified Jury table using numerator polynomial of original system until the array length becomes two. This is the reduced second order equation of numerator polynomial.

Step(ii): Similarly obtain the order reduction of denominator polynomial of original system to second order equation using the modified Jury table.

Step (iii) : Gain adjustment of the new reduced order system (R(z)) whose numerator and Denominator coefficients are obtained from the step (i) and step (ii) as follows:

The Steady state gain of original system G(z) is given by

$$\alpha_0 = G(z)\Big|_{(z=1)} = \frac{a_{n-1}+a_{n-2}+\cdots\ldots+a_2+a_1+a_0}{b_n+b_{n-1}+\cdots\ldots+b_2+b_1+b_0} \tag{4}$$

The steady state gain of reduced order model is obtained from (3)

$$\beta_0 = R(z)\Big|_{(z=1)} \frac{y_1+y_0}{x_2+x_1+x_0} \tag{5}$$

The gain factor (G.F) is given by G.F.= $\frac{\alpha 0}{\beta 0}$ (6)

The modified reduced order model $R_2(z)$ = (G.F) x R(z) is given by (7)

$$R_2(z) = (G.F) \times R(z) = \frac{y_1 z + y_0}{x_2 z^2 + x_1 z + x_0}$$

Formulation of Third Order Model

The third order model is formulated as follows: the coefficients of the numerator of the transfer function is obtained by using (2n-3)th row of the table I. The coefficients of the denominator of the transfer function is obtained using (2n-5)th row of the table II. And the reduced order model is given as

$$\mathbf{R_3(z)} = \frac{p_2 z^2 + p_1 z + p_0}{q_3 z^3 + q_2 z^2 + q_1 z + q_0} \tag{8}$$

Similarly the 4th , 5th……..(n-1)th order models may be constructed. These lower order models may be tested for stability and a suitable lower order model may be chosen for further analysis. The suitability of R(z) is analyzed by comparing the unit step response of R(z) with that of G(z).

Error Index

The deviation of lower order system from the original system is given by the cumulative error criterion j.

$$\mathbf{j} = \sum_{k=0}^{N} [\mathbf{y(k)} - \mathbf{y_r(k)}]^2 \tag{9}$$

where y(k) and $y_r(k)$ are the unit step responses of the original and reduced order discrete systems respectively and N is total time index. k = 1, 2,……..N. The best response corresponds to the minimum value of the error criterion.

3. Design of PID Controller using Reduced Order Model

Simplification of mathematical models of complex system is essential in engineering applications, also the computations involved in the design of PID controller for larger order stable system can be reduced with the help of suitable reduced order model. The basic structure of conventional feedback control system is shown in fig. 1. One of the widely used controller in the design of discrete systems is the PID controller. The transfer function of the digital PID controller is given as

$$\mathbf{Gc(z)} = \mathbf{k_p} + \mathbf{k_i}\frac{\mathbf{Tz}}{\mathbf{z-1}} + \mathbf{k_d}\frac{\mathbf{z-1}}{\mathbf{Tz}} \tag{10}$$

Where, k_p - is the proportional gain ; k_i - is the integral gain; k_d - is the derivative gain and T is the sampling time.

The design problem is to determine the values of the constants k_p , k_i, and k_d so that the performance of the system meets the desired specifications such as (i) over shoot < 3% (ii) settling time < 3 sec. and (iii) steady state error < 2%.

Using the proposed scheme, a second order model is obtained and a PID controller is designed for this reduced order model. The initial values of the parameters of PID controller are obtained by applying pole-zero cancellation method. The PID controller is attached with the reduced order model and the output response in observed. If the system response is not satisfied, then the parameters of the controller are tuned to obtain the response with desired specifications. The tuned controller is then attached with the original higher order system and the responses are observed.

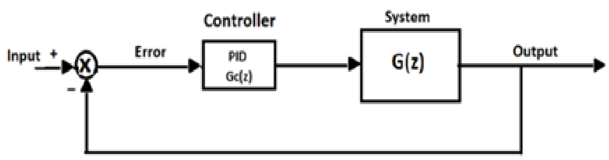

Fig. 1. Basic Feedback Control System with PID

4. Ilustrations

The stable Linear Time Invariant Discrete System given is $\frac{Z^3+0.9\,Z^2-0.01\,Z-0.009}{Z^4+0.6\,Z^2-0.0256}$ [8]

TABLE III. PROPOSED NUMERATOR ORDER REDUCTION

-0.009	- 0.01	0.9	1
1	0.9	- 0.01	-0.009
0.0019	-0.8999	-0.9999(∂1)	
-0.9999	-0.8999	0.0019	
-0.9015	-0.9998(∂2)		
-0.9998	-0.9015		

From table III, Reduced Numerator polynomial obtained is [-0.9998 Z- 0.9015]. The Stability of the reduced order polynomial can be checked by examining the δ values. From the table.3, δ1, δ2 are having negative signs and sufficient conditions are satisfied. Hence the reduced order numerator polynomial is stable.

TABLE IV. PROPOSED DENOMINATOR ORDER REDUCTION

-0.0256	0	0.6	0	1
1	0	0.6	0	-0.0256
0	-0.615	0	-0.99(∂1)	
-0.99	0	-0.615	0	
-0.6150	0	-0.9987(∂2)		
-0.9987	0	-0.6150		

From Table IV, reduced denominator polynomial obtained is [-0.9987 Z2- 0.6150] and the δ1, δ2 are having negative signs hence the reduced order denominator polynomial is stable.

From table III and table IV, the reduced order model

$$R(z) = \frac{[-0.9998\,Z - 0.9015]}{[-0.9987\,Z^2 - 0.6150]} = \frac{[0.9998\,Z + 0.9015]}{[\,Z^2 + 0.6158]}$$

Gain adjustment:

Original system steady state value = α_0 = 1.1947

Reduced order system steady state value = β_0 = 1.1768

Gain Factor (G.F) = 1.1947/1.1768 = 1.0153

Reduced order model after gain adjustment

$$R_1(z) = \frac{[1.0151\,Z + 0.9153]}{[\,Z^2 + 0.6158]}$$

• The responses of Original and reduced order systems are shown in fig.2. From the graph, it is inferred that the proposed method gives a cumulative error index of 0.0022 whereas Shamash model gives 0.32 [2].

• The response of the proposed reduced model is as close as to original system response

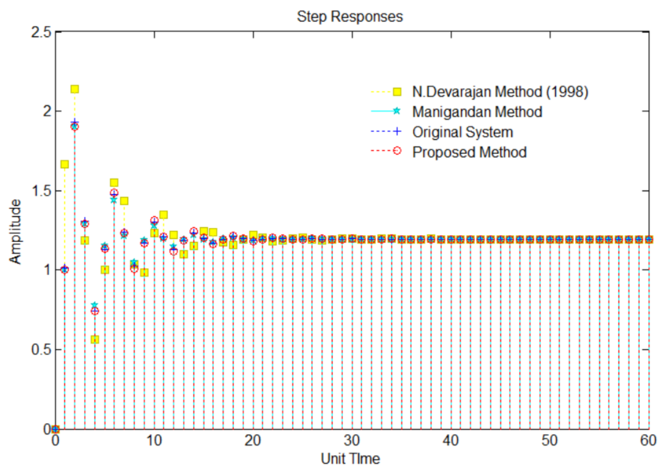

Fig. 2. Comparison of step responses

The proposed design is compared with the recent works in table V and fig.2. The ISE is calculated for 60 Unit Time with the increment of 1 Unit Time.

TABLE V. COMPARISON OF VARIOUS REDUCED ORDER MODELS

Sl.No.	Methods	ISE
1.	N.Devarajan Method (1998) [3] $\frac{\mathbf{1.6658328\,z-0.0601118}}{\mathbf{z^2-0.3196075\,z+0.6635985}}$	0.7043
2.	Manigandan Method [9] $\frac{\mathbf{z-0.9268}}{\mathbf{z^2+0.0269\,z+0.586}}$	0.0093
3.	Proposed Method $\frac{[\mathbf{1.0151\,Z+0.9153}]}{[\mathbf{Z^2+0.6158}]}$	0.0022

Important objective of order reduction is to design a controller to the reduced order system which would effectively control the original higher order system. As per section.3, the PID parameters with proposed reduced order system (after tuning) are Kp = 4.0674; Ki = 1.9320; Kd = 2.8079 with the sampling time 1. Then the above designed PID controller is connected to the original system. The corresponding system responses are obtained as shown in fig.3 and fig.4 from which, it is inferred that the controller designed for reduced order system works well when applied to the original system without any need of further tuning. The performance specifications are compared in table VI.

TABLE VI. ORIGINAL AND REDUCED ORDER SYSTEM PERFORMANCE WITH PID CONTROLLER

Sl.No	Specifications	R(z) with PID	G(z) with same PID
1.	Rise Time (unit time)	0	0
2.	Overshoot (%)	0	0.1083%
3.	Settling Time (unit time)	3	3
4.	Steady State Error	0	0

5. Discussion

From the preceding illustration, it is observed that the stability, initial and stead state value of original system is preserved by the proposed method. Any mismatch in the transient behavior can be overcome by adopting a soft computing technique which is not in the scope of this paper.

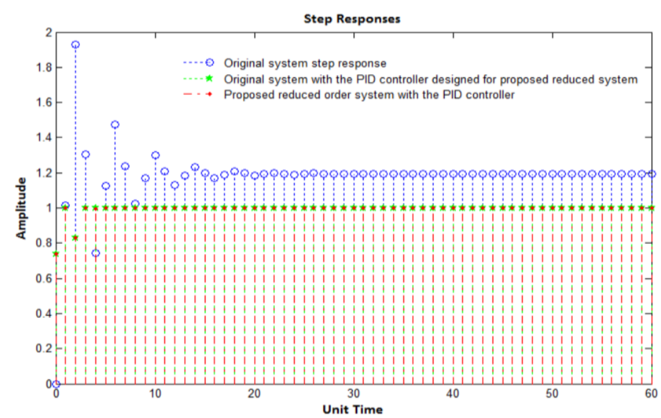

Fig. 3. Comparison of Step responses without and with PID

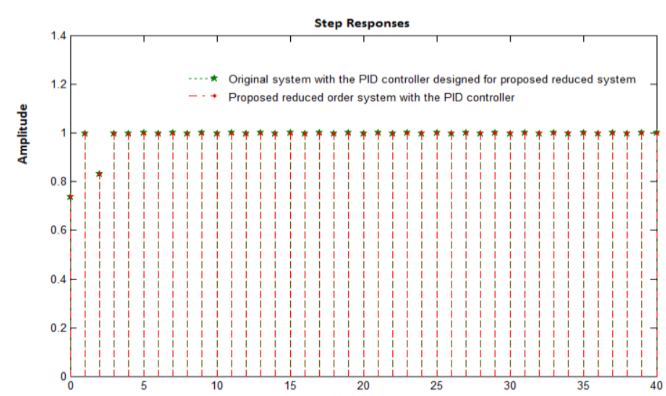

Fig. 4. Original and reduced order system step response with same PID

6. Conclusion

The proposed order reduction technique is based on the modified Jury table which unambiguously possesses the factor of stability of original system in each array which in turn assures the stability of the reduced order polynomial picked up from any of these arrays. Hence from the proposed modified Jury table, reduced order parameters are obtained directly for the numerator and denominator polynomials. This stability preserving reduced order system is also retaining the steady state value after the gain adjustment. The cumulative error between the original system and reduced order system responses is calculated. The proposed approach is comparable to the existing methods both in directness and complexity. To evaluate the proposed technique, a PID controller is designed and tuned for the reduced order system to meet the desired response and then tested with the original system which is found to give favorable results.

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