SCIENTIFIC BULLETIN
OF
THE POLITEHNICA UNIVERSITY OF TIMISOARA

Vol.69(83)	MATHEMATICS–PHISYCS	2024

In memory of Professor Borislav D. CRSTICI on the Centenary of his birth.

CLASSIFYING SPACES FOR FAMILIES OF SUBGROUPS FOR 8 -LOCATED GROUPS

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Abstract

We construct a low-dimensional classifying space for the family of virtually cyclic subgroups of a group acting properly on an 8-located complex with the SD'-property. The key property we use is that the minimal displacement set in an 8-located complex with the SD' property embeds isometrically into the complex and it is systolic. [1]

1 Introduction

Curvature can be expressed both in metric and combinatorial terms. On the metric side, one can refer to nonpositively curved in the sense of Aleksandrov and Gromov, i.e. by comparing small triangles in the space with triangles in the Euclidean plane. Such triangles must satisfy the CAT(0) inequality. On the combinatorial side, one can express curvature using a condition, called local 6-largeness which was introduced independently by Chepoi [3] (under the name of bridged complexes), Januszkiewicz-Świątkowski [11] and Haglund [9]. In [1], [4], [5], [6], [18], [12], [13] other conditions of this type are studied.

In [12] we study of a version of 8-location, suggested in [18, Subsection 5.1]. This 8-location says that homotopically trivial loops of length at most 8 admit filling diagrams with one internal vertex. However, in the new 8-location essential

[1]Mathematical Subject Classification (2010): {Primary 20F67, Secondary 05C99}
Keywords and phrases: simplicial complex, 8-location, SD' property, minimal displacement set, classifying spaces

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4-loops are allowed. In [12] (Theorem 4.3) it is shown that this local combinatorial condition is a negative-curvature-type condition. Namely, it is shown that simply connected, 8-located simplicial complexes are Gromov hyperbolic. In [10] we study 7-located complexes as defined initially in [18].

The purpose of the current paper is to construct a low-dimensional classifying space for the family of virtually cyclic subgroups of a group acting properly on an 8-located complex with the SD'-property. This construction relies on the structure of the minimal displacement set in an 8-located complex with the SD'-property. This set was studied for CAT(0) spaces (see [2]), for systolic complexes (see [7], [19]), for weakly systolic complexes (see [15]) and for 8-located complexes with the SD'-property (see [14]). Namely, for systolic, weakly systolic and 8-located complexes with the SD'-property, such set embeds isometrically into the complex and it is systolic. Therefore one can apply certain results proven in [7] and [19] on systolic complexes. As immediate consequence of these results, it follows that any isometry of an 8-located complex with the SD'-property either has a fixed simplex (elliptic case) or it stabilizes a thick geodesic (hyperbolic case). For systolic complexes, studying the structure of the minimal displacement set turns out to be useful when constructing a low-dimensional classifying space for the family of virtually cyclic subgroups of a group acting properly on a systolic complex (see [19]). We obtain similar results for 8-located simplicial complexes with the SD'-property.

2 Preliminaries

Let X be a simplicial complex. We denote by $X^{(k)}$ the k-skeleton of $X, 0 \leq k < \dim X$. A subcomplex L in X is called *full* as a subcomplex of X if any simplex of X spanned by a set of vertices in L, is a simplex of L. For a set $A = \{v_1, ..., v_k\}$ of vertices of X, by $\langle A \rangle$ or by $\langle v_1, ..., v_k \rangle$ we denote the *span* of A, i.e. the smallest full subcomplex of X that contains A. We write $v \sim v'$ if $\langle v, v' \rangle \in X$ (it can happen that $v = v'$). We write $v \nsim v'$ if $\langle v, v' \rangle \notin X$. We call X *flag* if any finite set of vertices which are pairwise connected by edges of X, spans a simplex of X.

A *cycle* (*loop*) γ in X is a subcomplex of X isomorphic to a triangulation of S^1. A *full cycle* in X is a cycle that is full as a subcomplex of X. A *k-wheel* in X $(v_0; v_1, ..., v_k)$ (where $v_i, i \in \{0, ..., k\}$ are vertices of X) is a subcomplex of X such that $\gamma = (v_1, ..., v_k)$ is a full cycle and $v_0 \sim v_1, ..., v_k$. The *length* of γ (denoted by $|\gamma|$) is the number of edges of γ. If $g = (v_1, ..., v_k)$ is a 1-skeleton geodesic of X, the *length* of g (denoted by $|g|$ or by $|(v_1, ..., v_k)|$) is the number of edges of g.

We define the *combinatorial metric* on the 0-skeleton of X as the number of edges in the shortest 1-skeleton path joining two given vertices.

A *ball (sphere)* $B_i(v, X)$ ($S_i(v, X)$) of radius i around some vertex v is a full subcomplex of X spanned by vertices at combinatorial distance at most i (at

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combinatorial distance i) from v.

Definition 2.1. *A simplicial complex is* m-located *if it is flag and every full homotopically trivial loop of length at most* m *is contained in a* 1*-ball.*

Let σ be a simplex of a simplicial complex X. The *link* of X at σ, denoted X_σ, is the subcomplex of X consisting of all simplices of X which are disjoint from σ and which, together with σ, span a simplex of X. We call a flag simplicial complex X, k*-large* if there are no full j-cycles in X, for $j < k$. We say X is *locally* k*-large* if all its links are k-large. We call X k-large, $k \geq 4$ if it is connected, simply connected and locally k-large.

We introduce further a global combinatorial condition on a flag simplicial complex.

Definition 2.2. *Let* X *be a flag simplicial complex. For a vertex* O *of* X *and a natural number* n*, we say that* X *satisfies* the property $SD'_n(O)$ *if for every* $i \in \{1, ..., n\}$ *we have:*

1. *(T) (triangle condition): for every edge* $e \in S_{i+1}(O)$*, the intersection* $X_e \cap B_i(O)$ *is non-empty;*

2. *(V) (vertex condition): for every vertex* $v \in S_{i+1}(O)$*, and for every two vertices* $u, w \in X_v \cap B_i(O)$*, there exists a vertex* $t \in X_v \cap B_i(O)$ *such that* $t \sim u, w$*.*

We say X *satisfies* the property $SD'(O)$ *if* $SD'_n(O)$ *holds for each natural number* n*. We say* X *satisfies* the property SD' *if* $SD'_n(O)$ *holds for each natural number* n *and for each vertex* O *of* X*.*

2.1 Minimal displacement set for systolic complexes

For systolic complexes the minimal displacement set is studied in [7].

Let h be an isometry of a simplicial complex X. We define the *displacement function* $d_h : X^{(0)} \to \mathbf{N}$ by $d_h(x) = d_X(h(x), x)$. The *translation length* of h is defined as $|h| = \min_{x \in X^{(0)}} d_h(x)$. If h does not fix any simplex of X, then h is called *hyperbolic*. In such case one has $|h| > 0$. Otherwise we call the isometry h *elliptic*. For a hyperbolic isometry h, we define the minimal displacement set $\mathrm{Min_X(h)}$ as the subcomplex of X spanned by the set of vertices where d_h attains its minimum. Clearly, $\mathrm{Min_X(h)}$ is invariant under the action of h.

Theorem 2.3. *Let* h *be a hyperbolic isometry of a systolic complex* X*. Then the subcomplex* $\mathrm{Min_X(h)}$ *is a systolic complex, isometrically embedded into* X *(see [7], Propositions* 3.3 *and* 3.4*).*

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Let h be an isometry of a simplicial complex X. An h-invariant geodesic in X is called an *axis* of h. We say that $\mathrm{Min_X(h)}$ is the union of axes, if for every vertex $x \in \mathrm{Min_X(h)}$, there is an h-invariant geodesic passing through x, i.e. $\mathrm{Min_X(h)}$ can be written as follows:

$$\mathrm{Min_X(h)} = \mathrm{span}\{\bigcup \gamma | \gamma \text{ is an } h\text{-invariant geodesic } \} \quad (2.1)$$

In this case, the isometry h acts on X as a translation along the axes by the number $|h|$.

For two subcomplexes $X_1, X_2 \subset X$, the distance $d_{\min}(X_1, X_2)$ is defined to be

$$d_{\min}(X_1, X_2) = \min\{d_X(x_1, x_2) | x_1 \in X_1, x_2 \in X_2\}.$$

We are ready now to define the *graph of axes*. For a hyperbolic isometry h satisfying (2.1), we define the simplicial graph Y_h as follows:

$$Y_h^{(0)} = \{\gamma | \gamma \text{ is an } h\text{-invariant geodesic in } \mathrm{Min_X(h)}\},$$
$$Y_h^{(1)} = \{\{\gamma_1, \gamma_2\} | d_{\min}(\gamma_1, \gamma_2) \leq 1\}.$$

Let $d_{Y(h)}$ denote the associated metric on $Y_h^{(0)}$.

2.2 Classifying spaces with finite or virtually cyclic stabilisers

The main goal of this section is, given a group G, to describe a method of constructing a model for a classifying space with virtually cyclic stabilisers out of a model for a classifying space with finite stabilisers. The presented method is due to W. Lück and M. Weiermann ([17]). First we give the necessary definitions.

A collection of subgroups $\mathcal{F}$ of a group G is called a *family* if it is closed under taking subgroups and conjugation by elements of G. Two examples which will be of interest to us are the family $\mathcal{FIN}$ of all finite subgroups, the family $\mathcal{VCY}$ of all virtually cyclic subgroups.

Definition 2.4. *Given a group G and a family of its subgroups $\mathcal{F}$, a* model for the classifying space $E_{\mathcal{F}}G$ *is a G-CW-complex X such that for any subgroup $H \subset G$ the fixed point set X^H is contractible if $H \in \mathcal{F}$, and empty otherwise.*

Let $\underline{E}G$ denote $E_{\mathcal{FIN}}G$ and let $\underline{\underline{E}}G$ denote $E_{\mathcal{VCY}}G$.

A model for $E_{\mathcal{F}}G$ exists for any group and any family. Any two models for $E_{\mathcal{F}}G$ are G-homotopy equivalent (see [16]). However, general constructions always produce infinite dimensional models.

We will describe a method of constructing a finite dimensional model for $\underline{\underline{E}}G$ out of a model for $\underline{E}G$ and appropriate models associated to infinite virtually cyclic subgroups of G. If $H \subset G$ is a subgroup and $\mathcal{F}$ is a family of subgroups of G, let $\mathcal{F} \cap H$ denote the family of all subgroups of H which belong to the family

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$\mathcal{F}$. More generally, if $\phi : H \to G$ is a homomorphism, let $\phi^\star\mathcal{F}$ denote the smallest family of subgroups of H that contains $\phi^{-1}(F)$ for all $F \in \mathcal{F}$.

Consider the collection $\mathcal{VCY}\setminus\mathcal{FIN}$ of infinite virtually cyclic subgroups of G. It is not a family since it does not contain the trivial subgroup. Define an equivalence relation on $\mathcal{VCY} \setminus \mathcal{FIN}$ by

$$H_1 \sim H_2 \iff |H_1 \cap H_2| = \infty$$

Let $[H]$ denote the equivalence class of H, and let $[\mathcal{VCY} \subset \mathcal{FIN}]$ denote the set of equivalence classes. The group G acts on $[\mathcal{VCY} \subset \mathcal{FIN}]$ by conjugation, and for a class $[H] \in [\mathcal{VCY} \subset \mathcal{FIN}]$ define the subgroup $N_G(H) \subseteq G$ to be the stabiliser of $[H]$ under this action, i.e.

$$N_G(H) = \{g \in G|\ |g^{-1}Hg \cap H| = \infty\}$$

The subgroup $N_G(H)$ is called the *commensurator* of H, since its elements conjugate H to the subgroup commensurable with H. For $[H] \in [\mathcal{VCY} \subset \mathcal{FIN}]$ define the family $\mathcal{G}[H]$ of subgroups of $N_G(H)$ as follows

$$\mathcal{G}[H] = \{K \subset G|\ K \in [\mathcal{VCY} \subset \mathcal{FIN}], [K] = [H]\} \cup \{K \in \mathcal{FIN} \cap N_G[H]\}.$$

Definition 2.5. *A group G satisfies* condition (C) *if for every $g, h \in G$ with $|h| = \infty$ (infinite order) and for any $k, l \in \mathbf{Z}$, we have*

$$gh^kg^{-1} = h^l \implies |k| = |l|$$

Lemma 2.6. *Let $K \subset N_G[H]$ be a finitely generated subgroup that contains some representative of $[H]$ and assume that the group G satisfies condition (C). Choose an element $h \in H$ such that $[\langle h\rangle] = [H]$ (any element of infinite order has this property). Then there exists an integer $k \geq 1$, such that $\langle h^k\rangle$ is normal in K.*

For the proof see [19], Lemma 2.6, page 8.

3 Minimal displacement set for 8 - located simplicial complexes with the SD'-property

The minimal displacement set for 8-located complexes with the SD'-property is studied in [14]. The main result obtained there is given below.

Theorem 3.1. *For a (simplicial) isometry h with no fixed simplices of an 8 - located simplicial complex X with the SD'-property, the 1-skeleton of $\mathrm{Min_X(h)}$ is isometrically embedded into X.*

For the proof see [14], Theorem 3.3.

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Theorem 3.2. *Let X be an 8-located simplicial complex with the SD'-property. Then the 1-skeleton of X, equipped with the standard path metric, is Gromov hyperbolic.*

For the proof see [14], Corollary 3.4.

Theorem 3.3. *Let h be a (simplicial) isometry with no fixed simplices of an 8-located complex X with the SD'-property. Then the set $\mathrm{Min_X(h)}$ is systolic.*

Proof. Due to Theorem 3.1, the 1-skeleton of $\mathrm{Min_X(h)}$ is isometrically embedded into X. Moreover, Theorem 3.2 implies that $\mathrm{Min_X(h)}$ is Gromov hyperbolic. Then $\mathrm{Min_X(h)}$ is 7-systolic and therefore systolic. □

The following results on 8-located complexes with the SD'-property are immediate consequences of the fact that the minimal displacement set of a nonelliptic isometry acting on such complex is systolic and it embeds isometrically into the complex. Their systolic analogues, also given below, imply these similarities.

Theorem 3.4. *Let h be a hyperbolic simplicial isometry of a uniformly locally finite systolic complex X. Then there is an h^n-invariant geodesic for some $n \geq 1$.*

For the proof see [7], Theorem 3.5, page 46.

Theorem 3.5. *Let h be a hyperbolic simplicial isometry of a uniformly locally finite 8-located complex X with the SD'-property. Then there is an h^n-invariant geodesic for some $n \geq 1$.*

Proof. Theorem 3.3 implies that $\mathrm{Min_X(h)}$ is systolic. Then, by Theorem 3.4, in $\mathrm{Min_X(h)}$ there is an h^n-invariant geodesic for some $n \geq 1$. By Theorem 3.1, $\mathrm{Min_X(h)}^{(1)}$ is isometrically embedded into X. Then this h^n-invariant geodesic also belongs to X. This completes the proof.

□

Theorem 3.6. *Let h be a simplicial isometry of a uniformly locally finite systolic complex X. Then either there is an h-invariant simplex (elliptic case) or there is an h-invariant thick geodesic (hyperbolic case).*

For the proof see [7], Theorem 3.8, page 49.

Theorem 3.7. *Let h be a simplicial isometry of a uniformly locally finite 8-located complex X with the SD'-property. Then either there is an h-invariant simplex (elliptic case) or there is an h-invariant thick geodesic (hyperbolic case).*

Proof. Let $Y = \mathrm{Min_X(h)}$. Theorem 3.3 implies that Y is systolic. Then, by Theorem 3.6, in Y either there is an h-invariant simplex (elliptic case) or there is an h-invariant thick geodesic (hyperbolic case). Since, by Theorem 3.1, $Y^{(1)}$ is isometrically embedded into X, either this h-invariant simplex or this h-invariant thick geodesic, also belongs to X.

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Theorem 3.8. *Let h be a hyperbolic simplicial isometry of a uniformly locally finite systolic complex X. If there exists an h^n-invariant geodesic in X, then for any vertex $x \in \mathrm{Min_X(h^n)} \subset \mathrm{X}$, there exists an h^n-invariant geodesic passing through x.*

For the proof see [7], Remark page 48.

Theorem 3.9. *Let h be a hyperbolic simplicial isometry of a uniformly locally finite 8-located complex X with the SD'-property. If there exists an h^n-invariant geodesic in X, then for any vertex $x \in \mathrm{Min_X(h^n)} \subset \mathrm{X}$, there exists an h^n-invariant geodesic passing through x.*

Proof. Let $Y = \mathrm{Min_X(h)}$. Theorem 3.3 implies that Y is systolic. According to Theorem 3.4, in Y (and then, by Theorem 3.5, also in X) there exists an h^n-invariant geodesic for some $n \geq 1$. Hence, by Theorem 3.8, for any vertex $x \in \mathrm{Min_X(h^n)} \subset \mathrm{Y}$, there exists an h^n-invariant geodesic passing through x. By Theorem 3.1, $Y^{(1)}$ is isometrically embedded into X. Then for any vertex $x \in \mathrm{Min_X(h^n)} \subset \mathrm{X}$, there exists an h^n-invariant geodesic passing through x. □

4 Classifying spaces with virtually cyclic stabiliser for 8-located groups with the SD'-property

In this section we construct a low-dimensional classifying space for the family of virtually cyclic subgroups of a group acting properly on an 8-located complex with the SD'-property.

Theorem 4.1. *Let h be a hyperbolic isometry of a uniformly locally finite systolic complex X such that the subcomplex $\mathrm{Min_X(h)}$ is the union of axes. Then there is a quasi-isometry $c : (Y(h) \times \mathbf{Z}, d_h) \to (\mathrm{Min_X(h)}, \mathrm{d_X})$ where the metric d_h is defined as $d_h((\gamma_1, t_1), (\gamma_2, t_2)) = d_{Y(h)}(\gamma_1, \gamma_2) + |t_1 - t_2|$, and d_X is the metric induced from X.*

For the proof see [19], Theorem 3.3, page 12.

Theorem 4.2. *Let h be a hyperbolic isometry of a uniformly locally finite 8-located complex X with the SD'-property such that the subcomplex $\mathrm{Min_X(h)}$ is the union of axes. Then there is a quasi-isometry $c : (Y(h) \times \mathbf{Z}, d_h) \to (\mathrm{Min_X(h)}, \mathrm{d_X})$ where the metric d_h is defined as $d_h((\gamma_1, t_1), (\gamma_2, t_2)) = d_{Y(h)}(\gamma_1, \gamma_2) + |t_1 - t_2|$, and d_X is the metric induced from X.*

Proof. Due to Theorem 3.1, the 1-skeleton of $\mathrm{Min_X(h)}$ is isometrically embedded into X. Moreover, Theorem 3.3 implies that $\mathrm{Min_X(h)}$ is systolic. Hence the result follows by Theorem 4.1. □

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Theorem 4.3. *Let X be a locally finite systolic simplicial complex. For a hyperbolic isometry h whose minimal displacement set is a union of axes (that is, for h satisfying* (2.1)*), the graph of axes $(Y(h), d_{Y(h)})$ is quasi-isometric to a simplicial tree.*

For the proof see [19], Corollary 4.6, page 21.

Theorem 4.4. *Let X be a locally finite 8-located complexes with the SD'-property. For a hyperbolic isometry h whose minimal displacement set is a union of axes (that is, for h satisfying* (2.1)*), the graph of axes $(Y(h), d_{Y(h)})$ is quasi-isometric to a simplicial tree.*

Proof. If there do not exist h-invariant geodesics in X, take an h^n-invariant geodesic in X, $n > 1$. Assume there exist h-invariant geodesics in X. These geodesics belong to $\mathrm{Min_X(h)}$ because, according to (2.1),

$$\mathrm{Min_X(h)} = \mathrm{span}\{\bigcup \gamma | \gamma \text{ is an } h\text{-invariant geodesic } \}.$$

Due to Theorem 3.1, the 1-skeleton of $\mathrm{Min_X(h)}$ is isometrically embedded into X. Moreover, Theorem 3.3 implies that $\mathrm{Min_X(h)}$ is systolic. Then, by Theorem 4.3, the desired result holds.

□

For the rest of the section, let G be a group acting properly on a uniformly locally finite 8-located complex X with the SD'-property of dimension d. Let h be a hyperbolic isometry of X.

Theorem 4.5. *The systolic complex X is a model for $\underline{E}G$.*

For the proof see [6], Theorem E.

In order to construct models for the commensurators $N_G[H]$, first we show that the group G satisfies Condition (C). Using this, in every finitely generated subgroup $K \subseteq N_G[H]$ that contains H we find a suitable normal cyclic subgroup. As shown in [19] for systolic complexes, the quotient group acts properly on a quasi-tree.

Lemma 4.6. *The group G satisfies condition (C).*

Proof. The proof is similar to the one given in [19] (Lemma 5.2). Take arbitrary $g, h \in G$ such that $|h| = \infty$, and assume there are $k, l \in \mathbf{Z}$ such that $g^{-1}h^k g = h^l$. We have to show that $|k| = |l|$. Since the action of G on X is proper, the element h acts as a hyperbolic isometry. By Theorem 3.5, in X there is an h^n-invariant geodesic for some $n \geq 1$. We get the claim by considering the following sequence of equalities for the translation length:

$$|k| \cdot |h^n| = |h^{k \cdot n}| = |g^{-1} \cdot h^{n \cdot k} \cdot g| = |h^{\pm l \cdot n}| = |l| \cdot |h^n|.$$

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The first and the last of the equalities follow from the fact that the translation length of an element can be measured on an invariant geodesic, the second one is an easy calculation and the third one is straightforward. □

Lemma 4.7. *Let K be a finitely generated subgroup of G, and let $h \in K$ a hyperbolic isometry satisfying* (2.1) *such that $\langle h \rangle$ is normal in K. Then the proper action of G on X induces a proper action of $K/\langle h \rangle$ on the graph of axes $Y(h)$.*

Proof. The proof is similar to the one given in [19], Lemma 5.3 in the systolic case. □

Lemma 4.8. *Let h be a hyperbolic isometry of an 8-located complex X with the SD'-property. If h satisfies* (2.1)*, then so does h^n for any $n \in \mathbf{Z} \setminus \{0\}$.*

Proof. The result follows due to the fact that an h-invariant geodesic is h^n-invariant. □

Lemma 4.9. *Let K be a finitely generated subgroup of $N_G[H]$ that contains H. Then there is a short exact sequence*

$$0 \to \langle h \rangle \to K \to K/\langle h \rangle \to 0,$$

such that $h \in H$ is of infinite order and the group $K/\langle h \rangle$ is virtually free.

Proof. The proof is similar to the one given in [19], Lemma 5.4. Choose an element of infinite order $\tilde{h} \in H$ satisfying the following two conditions:

1. the set $\mathrm{Min}(\tilde{h})$ is the union of axes (see (2.1));
2. the translation length $|\tilde{h}| > 3$.

Both conditions above can be ensured by rising $\tilde{h}$ to a sufficiently large power. Indeed, by Lemma 3.5, there exists $n \geq 1$ such that $\tilde{h}^n$ satisfies the first condition above. If $|\tilde{h}^n| \leq 3$, then replace it by $\tilde{h}^{4n}$. The element $\tilde{h}^{4n}$ satisfies both conditions. If an element satisfies the conditions above then, by Lemma 4.8, so does any of its powers. Since G satisfies Condition (C), by Lemma 2.6, there exists an integer $k \geq 1$ such that $\langle \tilde{h}^k \rangle$ is normal in K.

Put $h = \tilde{h}^k$. By Lemma 4.7, the group $K/\langle h \rangle$ acts properly by isometries on the graph of axes $(Y(h), d_{Y(h)})$, which is, by Theorem 4.4, a quasi-tree. In conclusion the group $K/\langle h \rangle$ is virtually free.

□

The proofs of the next results are similar to the one given for systolic complexes in [19] (see Lemma 5.6, Theorem C).

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Theorem 4.10. *For every* $[H] \in \mathcal{VCY} \setminus \mathcal{FIN}$ *there exist:*

1. *a* 2*-dimensional model for* $E_{\mathcal{Y}[H]}N_G[H]$*;*
2. *a* 3*-dimensional model for* $\underline{E}N_G[H]$*.*

Theorem 4.11. *There exists a model for* $\underline{\underline{E}}G$ *of dimension*

$$\dim \underline{\underline{E}}G = \begin{cases} d{+}1, & \text{if } d \leq 3, \\ d, & \text{if } d \geq 4. \end{cases}$$

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