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In memory of Professor Borislav D. CRSTICI on the Centenary of his birth.

ON THE STABILITY OF MULTIPLIERS ON BANACH ALGEBRAS

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Abstract

In this paper, we give new results concerning the Hyers-Ulam stability of multipliers on Banach algebras. So, we extend some results obtained in 2004 by T. Miura, G. Hirasawa and S.E. Takahasi.
[1]

1 Introduction

In 1940, S.M. Ulam (see [21]) raised the following problem:
given an approximately homomorphism, can one find a homomorphism near it?

In 1941, D.H. Hyers [7] gave an affirmative answer to the question of Ulam for addditive Cauchy equation in Banach spaces. For this reason, one says that the Cauchy equation is stable in the sense of Hyers-Ulam. Afterwards, different generalizations of that initial answer of Hyers were obtained. See, for example, the books [3], [8], [10] and the references therein. See also the recent papers of the author [2], [15] and [16].

The stability in the Hyers-Ulam sense of multipliers on the Banach algebra was considered for the first time by T. Miura, G. Hirasawa and S.E. Takahasi in 2004 (see [17]). See also the papers [1], [5], [12] and [18]. In the following, we will extend their results, by giving new results concerning the generalized stability in the sense of Hyers-Ulam of multipliers on Banach algebras.

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We denote by A a Banach algebra and let $T : A \to A$. We say that T is a multiplier on A if it satisfies condition

$$(I) \qquad aT(b) = T(ab) = T(a)b, \ (\forall)\ a, b \in A.$$

If T satisfies only the condition

$$(II) \qquad aT(b) = T(a)b, \ (\forall)\ a, b \in A,$$

we say that T is a weak multiplier.

In the paper [11], it is given an example of an operator T, which satisfies the condition (II), but it not satisfies (I). See also [12]. The theory of multipliers (centralizers) on Banach algebra was discussed in now classical papers [4], [9] and [20]. The first who investigated the theory of multipliers in Banach algebras was C. Foiaş, in connection with a certain problem of abstract harmonic analysis. See also the recent papers [6], [13], [19]. See also the book [14].

We say that the algebra A is without the left order if $yA = \{0\}$ implies $y = 0$.

We say that the algebra A is without the right order if $Ay = \{0\}$ implies $y = 0$.

2 Hyperstability of multipliers

In the following, we denote by A a Banach algebra, $T : A \to A$ and $\gamma : A \times A \to [0, \infty)$.

We prove a result on the hyperstability of multipliers. First, we give the following Lemma.

Lemma 2.1. *(Folklore) If $T : A \to A$ is a weak multiplier and A is without right order, then T is a multiplier.*

Proof. Let x be an arbitrary element of A. Since T is a weak multiplier, we have

$$xT(ab) = T(x)ab = xT(a)b.$$

It follows $x[T(ab) - T(a)b] = 0, \ (\forall)\ x \in A$.
Because A is without right order, we obtain $T(ab) = T(a)b = aT(b)$. □

Theorem 2.2. *We suppose that A is a Banach algebra without right order. We suppose that the following conditions holds:*

(i) $\|a(Tb) - (Ta)b\| \leq \gamma(a, b)$, *for all* $a, b \in A$;

(ii) $(\exists)(\lambda_n) \subset \mathbb{C} \setminus \{0\}$ *such that* $\lim\limits_{n\to\infty} \dfrac{\gamma(\lambda_n a, b)}{|\lambda_n|} = 0$, *for all* $a, b \in A$.

Then T is a multiplier of A.

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Proof. We prove that T is a homogeneous mapping. Let be $\lambda \in \mathbb{C}$ and $x \in A$. In (i), we take $a = \lambda_n x$ and $b = \lambda a$ to obtain:

$$\|\lambda_n x T(\lambda a) - T(\lambda_n x)\lambda a\| \leq \gamma(\lambda_n x, \lambda a)$$

We have

$$\begin{aligned}\|\lambda_n x T(\lambda a) - \lambda_n x \lambda T(a)\| &\leq \|\lambda_n x T(\lambda a) - T(\lambda_n x)\lambda a\| + \|T(\lambda_n x)\lambda a - \lambda_n x \lambda T(a)\| \\ &\leq \gamma(\lambda_n x, \lambda a) + |\lambda|\|T(\lambda_n x)a - \lambda_n x T(a)\| \\ &\leq \gamma(\lambda_n x, \lambda a) + |\lambda|\gamma(\lambda_n x, a)\end{aligned}$$

hence

$$\|xT(\lambda a) - x\lambda T(\lambda)\| \leq \frac{\gamma(\lambda_n a, \lambda a)}{|\lambda_n|} + |\lambda|\frac{\gamma(\lambda_n x, a)}{|\lambda_n|}.$$

We take $n \to \infty$ and use condition (ii) to obtain $x[T(\lambda a) - \lambda T(a)] = 0$.
Since A is without right order we obtain $T(\lambda a) = \lambda T(a)$.
We prove that T is a multiplier. Since T is homogeneous, we have

$$\begin{aligned}\|a(Tb) - (Ta)b\| &= \left\|\frac{1}{\lambda_n}[\lambda_n a T(b) - T(\lambda_n a)b]\right\| \\ &\leq \frac{1}{|\lambda_n|}\gamma(\lambda_n a, b)\end{aligned}$$

We take $n \to \infty$ to obtain $a(Tb) - (Ta)b = 0$. Hence T is a weak multiplier and we apply Lemma 2.1 to obtain that T is a multiplier. □

Corollary 2.3. *We suppose that A is a Banach algebra without right order and $T : A \to A$. We suppose that*

$$\|a(Tb) - (Ta)b\| \leq \gamma(a, b), \ (\forall)\ a, b \in A.$$

We consider fixed $s \in \{1, -1\}$. We suppose that there exists $r \in [0, 2^s)$ such that the following condition holds

$$\gamma(2^s a, b) \leq r \cdot \gamma(a, b).$$

Then T is multiplier.

Proof. We have $\gamma(2^{sn}a, b) \leq r^n \gamma(a, b)$ hence

$$\frac{\gamma(2^{sn}a, b)}{2^{sn}} \leq \left(\frac{r}{2^s}\right)^n \gamma(a, b) \to 0, \text{ as } n \to \infty$$

and we apply Theorem 2.2. □

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Remark 2.4. *If we take the particular case when*

$$\gamma(a,b) = \varepsilon\|a\|^p\|b\|^p,$$

with $p \geq 0,\ p \neq 1$, *we obtain the result of Theorem 1.1 in [17].*

For the case $p = 1$, we have an extension of a counter example from the paper [17].

Proposition 2.5. *Let* $A \neq \{0\}$ *be a Banach algebra. Then for all* $\varepsilon > 0$, *there exists*

$$f : A \to A$$

such that

$$\|af(b) - f(a)b\| \leq \varepsilon\|a\|\|b\|,\ (\forall)\ a, b \in A$$

and f *is not a multiplier.*

Proof. We take $f : A \to A$,

$$f(a) = \frac{\varepsilon\|a\|}{2(1+\|a\|)}a$$

We have

$$\begin{aligned}\|af(b) - f(a)b\| &= \left\|\frac{\varepsilon\|b\|}{2(1+\|b\|)}ab - \frac{\varepsilon\|a\|}{2(1+\|a\|)}ab\right\| \\ &\leq \left(\frac{\varepsilon\|b\|}{2(1+\|b\|)} + \frac{\varepsilon\|a\|}{2(1+\|b\|)}\right)\|ab\| \\ &\leq \varepsilon\|a\|\|b\|\end{aligned}$$

We prove that there exists $a, b \in A$ such that $af(b) \neq f(a)b$. If $af(b) = f(a)b$, $(\forall)a, b \in A$, we have

$$\frac{\varepsilon\|b\|}{2(1+\|b\|)}ab = \frac{\varepsilon\|a\|}{2(1+\|a\|)}ab.$$

We take norms

$$\frac{\|a\|\|b\|^2}{1+\|b\|} = \frac{\|a\|^2\|b\|}{1+\|a\|}$$

and for $a, b \neq 0$,

$$\frac{\|b\|}{1+\|b\|} = \frac{\|a\|}{1+\|a\|}.$$

It follows $\|b\| + \|b\|\|a\| = \|a\| + \|a\|\|b\|$. Hence $\|a\| = \|b\|$, $(\forall)a, b \neq 0$, which is absurd. □

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3 Generalized Hyers-Ulam-Rassias stability of multipliers

Let A be a Banach algebra. Now we do not assume that A is without right order.

Theorem 3.1. *Let A be a Banach algebra, $(\lambda_n) \subset \mathbb{C} \setminus \{0\}$ and $f : A \to A$ be such that*

$$(iii)\ \Phi(a) := \sum_{n=0}^{\infty} \left\| \frac{f(\lambda_{n+1}a)}{\lambda_{n+1}} - \frac{f(\lambda_n a)}{\lambda_n} \right\| < \infty,\ (\forall)a \in A;$$

and

$$(iv)\ \|af(b) - f(a)b\| \leq \gamma(a,b), (\forall)a, b \in A$$

where $\gamma : A \times A \to [0, \infty)$ is such that

$$(v)\ \lim_{n\to\infty} \frac{\gamma(\lambda_n a, \lambda_n b)}{|\lambda_n|^2} = 0$$

If $\lambda_0 = 1$, then there exists a weak multiplier T of A such that

$$\|f(a) - T(a)\| \leq \Phi(a),\ a \in A.$$

Proof. Let a be a fixed element of A. From (iii) it follows hat for all $\varepsilon > 0$, there exists $N(\varepsilon)$ such that

$$\sum_{k=n}^{n+p} \left\| \frac{f(\lambda_{k+1}a)}{\lambda_{k+1}} - \frac{f(\lambda_k a)}{\lambda_k a} \right\| < \varepsilon,\ \text{for } n \geq N(\varepsilon),\ p \geq 1.$$

It follows

$$\left\| \frac{f(\lambda_{n+p+1}a)}{\lambda_{n+p+1}} - \frac{f(\lambda_n a)}{\lambda_n} \right\| < \varepsilon.$$

Hence $\left\{ \dfrac{f(\lambda_n a)}{\lambda_n} \right\}$ is a Cauchy sequence.

Since A is a Banach algebra, we deduce that this sequence converges. We denote by

$$T(a) := \lim_{n\to\infty} \frac{f(\lambda_n a)}{\lambda_n}.$$

We have

$$\|\lambda_n a f(\lambda_n b) - f(\lambda_n a)\lambda_n b\| \leq \gamma(\lambda_n a, \lambda_n b)$$

hence

$$\left\| a\frac{f(\lambda_n b)}{\lambda_n} - \frac{f(\lambda_n a)}{\lambda_n} b \right\| \leq \frac{\gamma(\lambda_n a, \lambda_n b)}{|\lambda_n|^2}$$

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We take $n \to \infty$ to obtain

$$\|aT(b) - T(a)b\| = 0$$

hence T is a weak multiplier.

From (iii), we have

$$\sum_{k=0}^{n-1} \left\| \frac{f(\lambda_{k+1}a)}{\lambda_{k+1}} - \frac{f(\lambda_k a)}{\lambda_k} \right\| \leq \Phi(a),\ n \geq 1.$$

hence

$$\left\| \frac{f(\lambda_n a)}{\lambda_n} - f(a) \right\| \leq \Phi(a),\ n \geq 1. \tag{1}$$

We take $n \to \infty$ to obtain $\|T(a) - f(a)\| \leq \Phi(a),\ a \in A$. □

Corollary 3.2. *Let be $s \in \{1, -1\}$. We suppose that $\psi : [0, \infty) \to [0, \infty)$ and there exists $r \in [0, 2^s)$ such that*

$$\psi(2^s t) \leq r\psi(t),\ t \in [0, \infty).$$

We suppose (iv) and (v) take place, where $\gamma : A \times A \to [0, \infty)$ and the following condition holds

(vi) $\|f(2^s a) - 2^s f(a)\| \leq \psi(\|a\|), (\forall)\ a \in A$.

Then there exists a weak multiplier $T : A \to A$ such that

$$\|f(a) - T(a)\| \leq \frac{1}{2^s - r}\psi(\|a\|),\ a \in A.$$

From (vi), we have $\|f(2^{s(n+1)}a) - 2f(2^{sn}a)\| \leq r^n\psi(\|a\|)$ hence

$$\Phi(a) := \sum_{n=0}^{\infty} \left\| \frac{f(2^{s(n+1)}a)}{2^{s(n+1)}} - \frac{f(2^{sn}a)}{2^{sn}} \right\| \leq \frac{1}{2^s} \sum_{n=0}^{\infty} \left(\frac{r}{2^s} \right)^n \psi(\|a\|) = \frac{1}{2^s - r}\psi(\|a\|)$$

and we apply Theorem 3.1.

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