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In memory of Professor Borislav D. CRSTICI on the Centenary of his birth.

STRONG DICHOTOMY WITH GROWTH RATES FOR LINEAR DISCRETE TIME SYSTEMS IN BANACH SPACES

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Abstract

The purpose of the this paper is to present characterizations of Datko-type for the concepts of strong h-dichotomy and h-dichotomy, are given using both invariant and strongly invariant projections sequences, for linear discrete time systems in Banach spaces. Also, as consequences we obtain characterizations for uniform h-dichotomy, strong eponential/polinomyal dichotomy and exponential/polynomial dichotomy. [1]

1 Introduction

Nowadays, it is required a discrete-time approach because the phenomena of the real world, in different domain takes place in a certain moment of time. The notion of exponential dichotomy introduced by Perron for differential equations and by Li for difference equations an dynamical systems.
In some situations, particularly in the nonautonomous settings, the concept of uniform exponential dichotomy is too restrictive and it is important to consider more general behaviors. One of the main reasons for weakening the assumption of uniform exponential dichotomy is that from the point of view of ergodic theory almost all variational equations in a finite-dimensional space admit a nonuniform

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exponential dichotomy. On the other hand it is important to treat the case of noninvertible systems because of their interest in applications (e.g., random dynamical systems, generated by random parabolic equations, are not invertible). Lately, characterizations of the nonuniform exponential dichotomy for discrete linear systems can be found in the works [3], [23], [28], [27] [26], [18], [19], [21], [30]. Characterizations of the nonuniform exponential dichotomy for discrete linear systems can be found in the works [5], [25], [22],[6], [18], [20].
Later, the concept of polynomial dichotomies was studied by Boruga, Seimeanu, Crai, Dragicevic and Sasu, Yue in works [7], [11], [13], [14], [29], [31].
The notion of dichotomy with growth rates in discrete-time was studied in [1], [8], [9].
In this paper we obtain different characterizations of Datko type for the dichotomy and strong dichotomy with growth rates for linear discrete time systems in Banach spaces with respect to invariant and strongly invariant projections sequences.

2 Preliminaries

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded operators from X into itself. The norms of both these spaces will be denote by $||\cdot||$. Let $\mathbb{N}$ be the set of all positive intergers and we deonte by Δ and T the follwing sets

$\Delta = \{(m,n) \in \mathbb{N}^2 : m \geq n\} \quad T = \{(m,n,p) \in \mathbb{N}^3 : m \geq n \geq p\}$.

In this paper we consider linear discrete-time systems of the form

$$(\mathcal{A}) \qquad x_{n+1} = A_n x_n, \quad n \in \mathbb{N}$$

where (A_n) is a sequence in $\mathcal{B}(X)$.
Then every solution $x = (x_n)$ of system $(\mathcal{A})$ is given by

$$x_m = A_m^n x_n, \quad for\ all\ (m,n) \in \Delta,$$

where

$$A_m^n = \begin{cases} A_{m-1}A_{m-2}...A_n, & m > n \\ I, & m = n \end{cases}$$

and I is the identity operator on X.

Remark 2.1. *We have the following properties:*
(i) $A_{n+1}^n = A_n$, *for all* $n \in \mathbb{N}$
(ii) $A_m^n A_n^p = A_m^p$, *for all* $(m,n,p) \in T$.

Definition 2.2. *A nondecreasing sequence* (h_n) *on* $[1,\infty)$ *is called growth rate sequence if* $\lim_{n\to\infty} h_n = \infty$.

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Definition 2.3. *A sequence (P_n) on $\mathcal{B}(X)$ is called projections sequence on X if*

$$P_n^2 = P_n, \quad for\ all\ n \in \mathbb{N}$$

Remark 2.4. *If (P_n) is projections sequence on X, then the sequence $Q_n = I - P_n$ is also a projections sequence on X, called the complementary projections sequence of P_n with $KerQ_n = RangeP_n$ and $RangeQ_n = KerP_n$ for every $n \in \mathbb{N}$.*

Definition 2.5. *The sequence (P_n) is invariant for the linear system $(\mathcal{A})$, if*

$$A_n P_n = P_{n+1} A_n, \quad for\ all\ n \in \mathbb{N}$$

Remark 2.6. *If (P_n) is invariant for $(\mathcal{A})$ then*

$$A_m^n P_n = P_m A_m^n \quad A_m^n Q_n = Q_m A_m^n,$$

for all $(m, n) \in \Delta$.

Definition 2.7. *The sequence (P_n) is strongly invariant for the linear system $(\mathcal{A})$ if it is invariant for $(\mathcal{A})$ and the restriction of A_m^n is an isomorphism from Range Q_n to Range Q_m.*

Remark 2.8. *If the sequence of projections (P_n) is a strongly invariant for the system $(\mathcal{A})$, exists $B : \Delta \to \mathbb{R}$ with $B(n, m) = B_n^m$: Range $Q_m \to$ Range Q_n isomorphism with $A_m^n B_n^m Q_m = Q_m$ and $B_n^m Am^n Q_n = Q_n$.*

Remark 2.9. *If (P_n) is a strongly invariant projections sequence for $(\mathcal{A})$ then there exists $B(n, m)$ as in the Remark 1.2.4 with the following properties:*

1. $B_n^m Q_m = Q_n B_n^m Q_m$
2. $Q_m = B_m^m Q_m = Q_m B_m^m Q_m$
3. $B_p^m Q_m = B_p^n B_n^m Q_m$

for all $(m, n), (n, p) \in \Delta$.

Proof. 1. For all $(m, n, x) \in \Delta \times X$ we have

$$Q_m x \in ImQ_m = KerP_m$$

so

$$B_m^n Q_m x \in KerP_n x = ImQ_m$$

then we have

$$B_n^m Q_m = B_n^m A_m^n B_n^m Q_m =$$
$$= B_n^m A_m^n Q_n B_n^m Q_m = Q_n B_n^m Q_m.$$

2. From Remark 4 we have that

$$Q_m = A_m^n B_n^m Q_m$$

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For $n = m$ we have

$$A_m^m B_m^m Q_m = B_m^m Q_m = Q_m.$$

3.

$$B_p^n B_n^m Q_m = B_p^n B_n^m A_m^n B_n^m Q_m =$$
$$= B_p^n B_n^m A_m^n A_n^p Q_n B_n^m Q_m = B_p^n B_n^m A_m^n Q_n A_n^p B_n^m Q_m =$$
$$= B_p^n Q_n A_n^p B_n^m Q_m = B_p^n A_n^p Q_n B_n^m Q_m =$$
$$= Q_n B_n^m Q_m = B_n^m Q_m.$$

□

3 The growth concepts for dichotomy with growth rates

Definition 3.1. *If the pair $(\mathcal{A}, P)$ has h-growth, then there are $M \geq 1, \omega > 0$ and $\delta \geq 0$ then we have:*

(hg_1) $h_n^\omega ||A_m^p P_p x|| \leq M h_m^\omega \frac{h_{n+1}}{h_n} h_n^\delta ||A_n^p P_p x||$

(hg_2) $h_n^\omega ||A_n^p Q_p x|| \leq M h_m^\omega \frac{h_{m+1}}{h_m} h_m^\delta ||A_m^p Q_p x||$

for all $(m, n, p, x) \in T \times X$.

Remark 3.2. *The pair $(\mathcal{A}, P)$ has h-growth if and only if there are $M \geq 1, \omega > 0$ and $\delta \geq 0$ with:*

$(hg_1^{'})$ $h_n^\omega ||A_m^n P_n x|| \leq M h_m^\omega \frac{h_{n+1}}{h_n} h_n^\delta ||P_n x||$

$(hg_2^{'})$ $h_n^\omega ||Q_n x|| \leq M h_m^\omega \frac{h_{m+1}}{h_m} h_m^\delta ||A_m^n Q_n x||$

for all $(m, n, x) \in \Delta \times X$.

Proof. Necessity. Results from Definition 5 for $p = n$.

Sufficiency. For $(hg_1') \implies (hg_1)$ we have

$$||A_m^p P_p x|| = ||A_m^n A_n^p P_p x|| \leq M \left(\frac{h_m}{h_n}\right)^\omega h_n^\delta \frac{h_{n+1}}{h_n} ||A_n^p P_p x||$$

for all $(m, n, p, x) \in T \times X$.

For $(hg_2') \implies (hg_2)$ we have

$$||A_m^p Q_p x|| = ||A_m^n A_n^p Q_p x|| \geq \frac{1}{M} \left(\frac{h_n}{h_m}\right)^\omega h_m^{-\delta} \frac{h_m}{h_{m+1}} ||A_n^p Q_p x||$$

for all $(m, n, p) \in T \times X$. □

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Remark 3.3. *The particular cases for the concept of h-growth are:*
1. *If $\delta = 0$, then we obtain the concept of uniform h-growth.*
2. *If $h_m = e^m$, we have exponential growth.*

Definition 3.4. *If the pair $(\mathcal{A}, P)$ has strong h-growth (s.h.g) then there are $M \geq 1, \omega > 0$ and $\delta \geq 0$ such that:*
(shg_1) $h_n^{\omega}||A_m^p P_p x|| \leq M h_m^{\omega} h_n^{\delta}||A_n^p P_p x||$
(shg_2) $h_n^{\omega}||A_n^p Q_p x|| \leq M h_m^{\omega} h_m^{\delta}||A_m^p Q_p x||$
for all $(m, n, p, x) \in \Delta \times X$.

Remark 3.5. *The pair $(\mathcal{A}, P)$ has strong h-growth if and only if there are $M \geq 1$, $\omega > 0$ and $\delta \geq 0$ such that:*
$(shg_1^{'})$ $h_n^{\omega}||A_m^n P_n x|| \leq M h_m^{\omega} h_n^{\delta}||P_n x||$
$(shg_2^{'})$ $h_n^{\omega}||Q_n x|| \leq M h_m^{\omega} h_m^{\delta}||A_m^n Q_n x||$
for all $(m, n, x) \in \Delta \times X$.

Proof. Necessity: Results from Definition 6 for $p = n$.
Sufficiency: For $(shg_1') \implies (shg_1)$ we have

$$h_n^{\omega}||A_m^p P_p x|| = h_n^{\omega}||A_m^n A_n^p P_p x|| \leq$$

$$\leq M h_m^{\omega} h_n^{\delta}||A_n^p P_p x||$$

and for $(shg_2') \implies (shg_2)$ we have

$$||A_m^p Q_p x|| = ||A_m^n A_n^p Q_p x|| \geq \frac{1}{M}\left(\frac{h_n}{h_m}\right)^{\omega} h_m^{-\delta}||A_n^p Q_p x||$$

for all $(m, n, p) \in T \times X$. □

Remark 3.6. *The particular cases for the concept of strong h-growth are:*
1. *If $\delta = 0$, then we obtain the concept of strong uniform h-growth.*
2. *If $h_m = e^m$, we have strong exponential growth.*

Proposition 3.7. *Let (P_n) strong invariant. The pair $(\mathcal{A}, P)$ has strong h-growth if and only if there are $M \geq 1, \omega > 0$ and $\delta \geq 0$ with:*
$(hg_1^{s''})$ $h_n^{\omega}||A_m^n P_n x|| \leq M h_m^{\omega} h_n^{\delta}||P_n x||$
$(hg_2^{s''})$ $h_n^{\omega}||B_n^m Q_m x|| \leq M h_m^{\omega} h_m^{\delta}||Q_m x||$
for all $(m, n, x) \in \Delta \times X$.

Proof. Neccessity. The implication $(hg_1^{'}) \implies (hg_1^{s'})$ is immediate for Remark 6. We will to prove $(hg_2^{'}) \implies (hg_2^{s'})$. From $(hg_2^{'})$ and Remark 5 we have

$$h_n^{\omega}||Q_n B_n^m Q_m x|| = h_n^{\omega}||Q_n B_n^m Q_m x|| \leq$$

$$\leq M h_m^{\omega} h_m^{\delta}||A_m^n Q_n B_n^m Q_m x|| \leq M h_m^{\omega} h_m^{\delta}||Q_m A_m^n B_n^m Q_m x|| \leq$$

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$$\leq Mh_m^{\omega}h_m^{\delta}||Q_mx||$$

for all $(m,n,x)\in\Delta\times X$.

Sufficiency. The implication $(hg_1^{s'})\implies(hg_1^{'})$ is immediate for Remark 6. We will to prove $(hg_2^{s'})\implies(hg_2^{'})$. From $(hg_2^{s'})$ and Remark 5 we have

$$h_n^{\omega}||Q_nx||=h_n^{\omega}||B_n^mA_m^nQ_nx||\leq Mh_m^{\omega}h_m^{\delta}||A_m^nQ_nx||$$

for all $(m,n,x)\in\Delta\times X$. □

Proposition 3.8. *e Let (P_n) is strong invariant, the pair $(\mathcal{A},P)$ has h-growth if and only if there are $M\geq 1,\omega>0$ and $\delta\geq 0$ with:*

$(shg_1^{s'})$ $h_n^{\omega}||A_m^nP_nx||\leq Mh_m^{\omega}\dfrac{h_{n+1}}{h_n}h_n^{\delta}||P_nx||$

$(shg_2^{s'})$ $h_n^{\omega}||B_n^mQ_mx||\leq Mh_m^{\omega}\dfrac{h_{m+1}}{h_m}h_m^{\delta}||Q_mx||$

for all $(m,n,x)\in\Delta\times X$.

Proof. Neccessity. The implication $(shg_1^{'})\implies(shg_1^{s'})$ is immediate for Remark 8. We will to prove $(shg_2^{'})\implies(shg_2^{s'})$. From $(shg_2^{'})$ and Remark 5 we have

$$h_n^{\omega}||Q_nB_n^mQ_mx||=h_n^{\omega}||Q_nB_n^mQ_mx||\leq$$

$$\leq Mh_m^{\omega}h_m^{\delta}\frac{h_{m+1}}{h_m}||A_m^nQ_nB_n^mQ_mx||\leq Mh_m^{\omega}h_m^{\delta}\frac{h_{m+1}}{h_m}||Q_mA_m^nB_n^mQ_mx||\leq$$

$$\leq Mh_m^{\omega}h_m^{\delta}\frac{h_{m+1}}{h_m}||Q_mx||$$

for all $(m,n,x)\in\Delta\times X$.

Sufficiency. The implication $(shg_1^{s'})\implies(shg_1^{'})$ is immediate for Remark 8. We will to prove $(shg_2^{s'})\implies(shg_2^{'})$. From $(shg_2^{s'})$ and Remark 4 we have

$$h_n^{\omega}||Q_nx||=h_n^{\omega}||B_n^mA_m^nQ_nx||\leq Mh_m^{\omega}h_m^{\delta}\frac{h_{m+1}}{h_m}||A_m^nQ_nx||$$

for all $(m,n,x)\in\Delta\times X$. □

4 The concepts of h-dichotomy with growth rates

Definition 4.1. *If then the pair $(\mathcal{A},P)$ is h-dichotomic (h.d.) then there are $N\geq 1,\epsilon\geq 0$ and $\nu>0$ such that:*

(hd$_1$) $h_m^{\nu}||A_m^pP_px||\leq Nh_n^{\nu}h_n^{\epsilon}\dfrac{h_{n+1}}{h_n}||A_n^pP_px||$

(hd$_2$) $h_m^{\nu}||A_n^pQ_px||\leq Nh_n^{\nu}h_m^{\epsilon}\dfrac{h_{m+1}}{h_m}||A_m^pQ_px||$

for all $(m,n,p,x)\in T\times X$.

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Remark 4.2. *The pair (A,P) is h-dichotomic if and only if there are $N\geq 1$, $\nu>0$ and $\epsilon\geq 0$ such that:*

$(hd_1^{'})$ $h_m^{\nu}||A_m^nP_nx||\leq Nh_n^{\nu}h_n^{\epsilon}\dfrac{h_{n+1}}{h_n}||P_nx||$

$(hd_2^{'})$ $h_m^{\nu}||Q_nx||\leq Nh_n^{\nu}h_m^{\epsilon}\dfrac{h_{m+1}}{h_m}||A_m^nQ_nx||$

for all $(m,n,x)\in\Delta\times X$.

Proof. Necessity. Results from Definition 7 for $p=n$.

Sufficiency. For $(shd_1')\implies(hd_1)$ we have

$$h_m^{\nu}||A_m^pP_px||=||A_m^nA_n^pP_px||\leq Nh_n^{\nu}h_n^{\epsilon}\frac{h_{n+1}}{h_n}||A_n^pP_px||$$

for all $(m,n,p,x)\in T\times X$.

For $(hd_2')\implies(hd_2)$ we have

$$||A_m^pQ_px||=||A_m^nA_n^pQ_px||\geq\frac{1}{N}\left(\frac{h_m}{h_n}\right)^{\nu}h_m^{-\epsilon}\frac{h_m}{h_{m+1}}||A_n^pQ_px||$$

for all $(m,n,p)\in T\times X$. □

Remark 4.3. *1. If $\epsilon=0$, we obtain the concept of uniform h-dichotomy (u.h.d.).*

2. If $h_n=e^n$, for all $n\in\mathbb{N}$ we obtain the concept of exponential dichotomy (e.d.)

3. If $h_n=n+1$, for all $m\in\mathbb{N}$ we obtain the concept polynomial dichotomy (p.d.)

4. If $h_n=\sqrt{n^2+1}$, for all $m\in\mathbb{N}$ we obtain the concept dichotomy with radical.

Definition 4.4. *If then the pair $(\mathcal{A},P)$ is strongly h-dichotomic (h.d.) then there are $N\geq 1,\epsilon\geq 0$ and $\nu>0$ such that:*

(shd_1) $h_m^{\nu}||A_m^pP_px||\leq Nh_n^{\nu}h_n^{\epsilon}||A_n^pP_px||$

(shd_2) $h_m^{\nu}||A_n^pQ_px||\leq Nh_n^{\nu}h_m^{\epsilon}||A_m^pQ_px||$

for all $(m,n,p,x)\in T\times X$.

Remark 4.5. *The pair (A,P) is strongly h-dichotomic if and only if there are $N\geq 1$, $\nu>0$ and $\epsilon\geq 0$ such that:*

$(shd_1^{'})$ $h_m^{\nu}||A_m^nP_nx||\leq Nh_n^{\nu}h_n^{\epsilon}||P_nx||$

$(shd_2^{'})$ $h_m^{\nu}||Q_nx||\leq Nh_n^{\nu}h_m^{\epsilon}||A_m^nQ_nx||$

for all $(m,n,x)\in\Delta\times X$.

Proof. Necessity. Results from Definition 8 for $p=n$.

Sufficiency. For $(shd_1')\implies(shd_1)$ we have

$$h_m^{\nu}||A_m^pP_px||=h_m^{\nu}||A_m^nA_n^pP_px||\leq Nh_n^{\nu}h_n^{\epsilon}||A_n^pP_px||$$

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for all $(m,n,p,x)\in T\times X$.

For $(shd_2^{'})\implies(hd_2)$ we have

$$||A_m^pQ_px||=||A_m^nA_n^pQ_px||\geq\frac{1}{N}\left(\frac{h_m}{h_n}\right)^{\nu}h_m^{-\epsilon}||A_n^pQ_px||$$

for all $(m,n,p)\in T\times X$. □

Remark 4.6. *1. If $\epsilon=0$, we obtain the concept of strong uniform h-dichotomy (s.u.h.d.).*

2. If $h_n=e^n$, for all $n\in\mathbb{N}$ we obtain the concept of strong exponential dichotomy (s.e.d.)

3. If $h_n=n+1$, for all $m\in\mathbb{N}$ we obtain the concept strong polynomial dichotomy (s.p.d.)

4. If $h_n=\sqrt{n^2+1}$, for all $m\in\mathbb{N}$ we obtain the concept dichotomy with radical.

Proposition 4.7. *Let (P_n) strong invariant. The pair (A,P) is strongly h-dichotomic if and only if there are $N\geq1$, $\nu>0$ and $\epsilon\geq0$ such that:*

$(hd_1^{s^{''}})$ $h_m^{\nu}||A_m^nP_nx||\leq Nh_n^{\nu}h_n^{\epsilon}||P_nx||$

$(hd_2^{s^{''}})$ $h_m^{\nu}||B_m^nQ_mx||\leq Nh_n^{\nu}h_m^{\epsilon}||Q_mx||$

for all $(m,n,x)\in\Delta\times X$.

Proof. Neccessity. The implication $(hd_1^{'})\implies(hd_1^{s^{'}})$ is immediate for Remark 13. We will to prove $(hd_2^{'})\implies(hd_2^{s^{'}})$. From $(hd_2^{'})$ and Remark 5 we have

$$h_m^{\nu}||Q_nB_n^mQ_mx||=h_m^{\nu}||Q_nB_n^mQ_mx||\leq$$

$$\leq Nh_n^{\nu}h_m^{\epsilon}||A_m^nQ_nB_n^mQ_mx||\leq Nh_n^{\nu}h_m^{\epsilon}||Q_mA_m^nB_n^mQ_mx||\leq$$

$$\leq Nh_n^{\nu}h_m^{\epsilon}||Q_mx||$$

for all $(m,n,x)\in\Delta\times X$.

Sufficiency. The implication $(hd_1^{s^{'}})\implies(hd_1^{'})$ is immediate for Remark 13. We will to prove $(hd_2^{s^{'}})\implies(hd_2^{'})$. From $(hd_2^{s^{'}})$ and Remark 5 we have

$$h_m^{\nu}||Q_nx||=h_m^{\nu}||B_n^mA_m^nQ_nx||\leq Nh_n^{\nu}h_m^{\epsilon}||A_m^nQ_nx||$$

for all $(m,n,x)\in\Delta\times X$. □

Proposition 4.8. *Let (P_n) strong invariant. The pair (A,P) is h-dichotomic if and only if there are $N\geq1$, $\nu>0$ and $\epsilon\geq0$ such that:*

$(shd_1^{s^{''}})$ $h_m^{\nu}||A_m^nP_nx||\leq Nh_n^{\nu}\dfrac{h_{n+1}}{h_n}h_n^{\epsilon}||P_nx||$

$(shd_2^{s^{''}})$ $h_m^{\nu}||B_n^mQ_mx||\leq Nh_n^{\nu}\dfrac{h_{m+1}}{h_m}h_m^{\epsilon}||Q_mx||$

for all $(m,n,x)\in\Delta\times X$.

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Proof. Neccessity. The implication $(shd_1^{'}) \Longrightarrow (shd_1^{s'})$ is immediate for Remark 11. We will to prove $(shd_2^{'}) \Longrightarrow (shd_2^{s'})$. From $(shd_2^{'})$ and Remark 5 we have

$$h_m^{\nu}||Q_nB_n^mQ_mx|| = h_m^{\nu}||Q_nB_n^mQ_mx|| \leq$$

$$\leq Nh_n^{\nu}h_m^{\epsilon}\frac{h_{m+1}}{h_m}||A_m^nQ_nB_n^mQ_mx|| \leq Nh_n^{\nu}h_m^{\epsilon}\frac{h_{m+1}}{h_m}||Q_mA_m^nB_n^mQ_mx|| \leq$$

$$\leq Nh_n^{\nu}h_m^{\epsilon}\frac{h_{m+1}}{h_m}||Q_mx||$$

for all $(m,n,x) \in \Delta \times X$.

Sufficiency. The implication $(shd_1^{s'}) \Longrightarrow (shd_1^{'})$ is immediate for Remark 11. We will to prove $(shd_2^{s'}) \Longrightarrow (shd_2^{'})$. From $(shd_2^{s'})$ and Remark 4 we have

$$h_m^{\nu}||Q_nx|| = h_m^{\nu}||B_n^mA_m^nQ_nx|| \leq Nh_n^{\nu}h_m^{\epsilon}\frac{h_{m+1}}{h_m}||A_m^nQ_nx||$$

for all $(m,n,x) \in \Delta \times X$. □

5 The main results

In what follows, we will give some characterizations of Datko type for dichotomy with growth rates for linear discrete time systems in Banach spaces with respect to invariant and strongly invariant projections sequences.

In order to do this, we will consider $\mathcal{H}$ the set of growth rates (h_n) that satisfy the following properties:

(1) $\exists H > 1 : h_{n+1} \leq Hh_n, \forall n \in \mathbb{N}$

(2) $\forall \alpha \in (-1,0),\ \exists\ H_1 > 1 : \sum_{k=m}^{\infty} h_k^{\alpha} \leq H_1 h_m^{\alpha},\ \forall m \in \mathbb{N}$

(3) $\forall \alpha \in (0,1),\ \exists\ H_2 > 1 : \sum_{j=0}^{m} h_j^{\alpha} \leq H_2 h_m^{\alpha},\ \forall m \in \mathbb{N}.$

Theorem 5.1. *The pair $(\mathcal{A}, P)$ is h-dichotomic if and only if there exist $D \geq 1, d > 0,\ \epsilon \geq 0$ with:*

(hD_1) $\sum_{k=n}^{\infty} \frac{h_{k+1}}{h_k} h_k^d ||A_k^pP_px|| \leq Dh_n^{\epsilon+d}||A_n^pP_px||$

(hD_2) $\sum_{j=n}^{m} \frac{h_{j+1}}{h_j} h_j^{-d} ||A_j^pQ_px|| \leq Dh_m^{\epsilon-d}||A_m^pQ_px||$

for all $(m,n,p,x) \in T \times X$.

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Proof. Necessity: Let be $d \in (0, \nu)$.
(hD_1)

$$\sum_{k=n}^{\infty} \frac{h_{k+1}}{h_k} h_k^d ||A_k^p P_p x|| \leq \sum_{k=n}^{\infty} \frac{h_{k+1}}{h_k} h_k^d N h_n^{\epsilon} \left(\frac{h_k}{h_n}\right)^{-\nu} \frac{h_{n+1}}{h_n} ||A_n^p P_p x|| \leq$$

$$\leq N H_1 h_n^{\epsilon} h_n^{\nu} ||A_n^p P_p x|| \sum_{k=n}^{\infty} \frac{h_{k+1}}{h_k} h_k^{d-\nu} \leq N H_1^2 H_2 h_n^{\epsilon} h_n^{\nu} h_n^{d-\nu} ||A_n^p P_p x|| \leq$$

$$\leq N H_1^2 H_2 h_n^{\epsilon+d} ||A_n^p P_p x|| = D h_n^{\epsilon+d} ||A_n^p P_p x||$$

where $D = N H_1^2 H_2 \geq 1$ and $(m, n, p, x) \in T \times X$.
(hD_2)

$$\sum_{j=n}^{m} \frac{h_{j+1}}{h_j} h_j^{-d} ||A_j^p Q_p x|| \leq \sum_{j=n}^{m} \frac{h_{j+1}}{h_j} h_j^{-d} N h_m^{\epsilon} \left(\frac{h_m}{h_j}\right)^{-\nu} \frac{h_{m+1}}{h_m} ||A_m^p Q_p x|| \leq$$

$$\leq N H_1 h_m^{\epsilon} h_m^{-\nu} ||A_m^p Q_p x|| \sum_{j=n}^{m} \frac{h_{j+1}}{h_j} h_j^{\nu-d} \leq N H_1^2 H_3 h_m^{\epsilon} h_m^{-\nu} h_m^{\nu-d} ||A_m^p Q_p x|| \leq$$

$$\leq N H_1^2 H_3 h_m^{\epsilon-d} ||A_m^p Q_p x|| = D h_m^{\epsilon-d} ||A_m^p Q_m x||$$

where $D = N H_1^2 H_3 \geq 1$ and $(m, n, p, x) \in T \times X$.

Sufficiency For $k = m$ in (hD_1) and $j = n$ in (hD_2), we obtain (hd_1), respectively (hd_2). □

Corollary 5.2. *The pair $(\mathcal{A}, P)$ is uniformly h-dichotomic if and only if there exist $D \geq 1, d > 0$, with:*

$$(uhD_1) \quad \sum_{k=n}^{\infty} \frac{h_{k+1}}{h_k} h_k^d ||A_k^p P_p x|| \leq D h_n^d ||A_n^p P_p x||$$

$$(uhD_2) \quad \sum_{j=n}^{m} \frac{h_{j+1}}{h_j} h_j^{-d} ||A_j^p Q_p x|| \leq D h_m^{-d} ||A_m^p Q_p x||$$

for all $(m, n, p, x) \in T \times X$.

Proof. Results immediate from Theorem 5 for $\epsilon = 0$. □

Corollary 5.3. *Let $h \in \mathcal{H}$. The pair $(\mathcal{A}, P)$ is exponentially dichotomic if and only if there are $D \geq 1, d > 0$, and $\epsilon \geq 0$ with:*

$$(eD_1) \quad \sum_{k=n}^{\infty} e^{dk} ||A_k^p P_p x|| \leq D e^{n(\epsilon+d)} ||A_n^p P_p x||$$

$$(eD_2) \quad \sum_{j=n}^{m} e^{-dj} ||A_j^p Q_p x|| \leq D e^{m(\epsilon-d)} ||A_m^p Q_p x||$$

for all $(m, n, p, x) \in T \times X$.

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Proof. It follows from Theorem 5 for $h_m = e^m$. □

Remark 5.4. *The preceding theorems are variants for the case of polynomial dichotomy property of well-known theorems due to Popa et al. [33,34] for exponential stability and exponential dichotomy.*

Theorem 5.5. *The pair $(\mathcal{A}, P)$ is strongly h-dichotomic if and only if there exist $D \geq 1, d > 0$, $\epsilon \geq 0$ with:*

$$(shD_1)\ \sum_{k=n}^{\infty} h_k^d ||A_k^p P_p x|| \leq D h_n^{\epsilon+d} ||A_n^p P_p x||$$

$$(shD_2)\ \sum_{j=n}^{m} h_j^{-d} ||A_j^p Q_p x|| \leq D h_m^{\epsilon-d} ||A_m^p Q_p x||$$

for all $(m, n, p, x) \in T \times X$.

Proof. Necessity. Let be $d \in (0, \nu)$.
(shD_1)

$$\sum_{k=n}^{\infty} h_k^d ||A_k^p P_p x|| \leq \sum_{k=n}^{\infty} h_k^d N h_n^{\epsilon} \left(\frac{h_k}{h_n}\right)^{-\nu} ||A_n^p P_p x|| \leq$$

$$\leq N h_n^{\epsilon} h_n^{\nu} ||A_n^p P_p x|| \sum_{k=n}^{\infty} h_k^{d-\nu} \leq N H_2 h_n^{\epsilon} h_n^{\nu} h_n^{d-\nu} ||A_n^p P_p x|| \leq$$

$$\leq N H_2 h_n^{\epsilon+d} ||A_n^p P_p x|| = D h_n^{\epsilon+d} ||A_n^p P_p x||$$

where $D = NH_2 \geq 1$ and $(m, n, p, x) \in T \times X$.
(shD_2)

$$\sum_{j=n}^{m} h_j^{-d} ||A_j^p Q_p x|| \leq \sum_{j=n}^{m} h_j^{-d} N h_m^{\epsilon} \left(\frac{h_m}{h_j}\right)^{-\nu} ||A_m^p Q_p x|| \leq$$

$$\leq N h_m^{\epsilon} h_m^{-\nu} ||A_m^p Q_p x|| \sum_{j=n}^{m} h_j^{\nu-d} \leq N H_3 h_m^{\epsilon} h_m^{-\nu} h_m^{\nu-d} ||A_m^p Q_p x|| \leq$$

$$\leq N H_3 h_m^{\epsilon-d} ||A_m^p Q_p x|| = D h_m^{\epsilon-d} ||A_m^p Q_m x||$$

where $D = NH_3 \geq 1$ and $(m, n, p, x) \in T \times X$.

Sufficiency. For $k = m$ in (shD_1) and $j = n$ in (shD_2) we obtain (shd_1) and (shd_2). □

Corollary 5.6. *The pair $(\mathcal{A}, P)$ is uniformly strongly h-dichotomic if and only if there exist $D \geq 1, d > 0$, with:*

$$(ushD_1)\ \sum_{k=n}^{\infty} h_k^d ||A_k^p P_p x|| \leq D h_n^d ||A_n^p P_p x||$$

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$$(ushD_2)\ \sum_{j=n}^{m} h_j^{-d}||A_j^p Q_p x|| \leq D h_m^{-d}||A_m^p Q_p x||$$

for all $(m, n, p, x) \in T \times X$.

Proof. Results immediate from Theorem 12 for $\epsilon = 0$. □

Corollary 5.7. *Let* $h \in \mathcal{H}$. *The pair* $(\mathcal{A}, P)$ *is strongly exponentially dichotomic if and only if there are* $D \geq 1, d > 0$, *and* $\epsilon \geq 0$ *with:*

$$(seD_1)\ \sum_{k=n}^{\infty} e^{dk}||A_k^p P_p x|| \leq D e^{n(\epsilon+d)}||A_n^p P_p x||$$

$$(seD_2)\ \sum_{j=n}^{m} e^{-dj}||A_j^p Q_p x|| \leq D e^{m(\epsilon-d)}||A_m^p Q_p x||$$

for all $(m, n, p, x) \in T \times X$.

Proof. It follows from Theorem 12 for $h_m = e^m$. □

In what follows, we will extend the characterizations obtained above, for the case of h- dichotomy and strong h-dichotomy with growth rates for discrete systems with respect to strongly invariant projections sequences. Therefore, next we will consider the linear system $(\mathcal{A})$ and (P_n) a projections sequences which is strongly invariant to $(\mathcal{A})$.

Theorem 5.8. *If the pair* $(\mathcal{A}, P)$ *is h-dichotomic then there exist* $D \geq 1, d > 0$, $\epsilon \geq 0$ *with:*

$$(hD_1^{s'})\ \sum_{k=n}^{\infty} \frac{h_{k+1}}{h_k} h_k^d||A_k^p P_p x|| \leq D h_n^{\epsilon+d}||A_n^p P_p x||$$

$$(hD_2^{s'})\ \sum_{j=n}^{m} \frac{h_{j+1}}{h_j} h_j^{-d}||B_j^m Q_m x|| \leq D h_m^{\epsilon-d}||Q_m x||$$

for all $(m, n, p, x) \in T \times X$.

Proof. Necessity. Let be $d \in (0, \nu)$.
The implication $(hd_1^{s'}) \implies (hD_1^{s'})$ is immediate from Theorem 12.
$(hd_2^{s'}) \implies (hD_2^{s'})$

$$\sum_{j=n}^{m} \frac{h_{j+1}}{h_j} \cdot h_j^{-d}||B_j^m Q_m x|| = \sum_{j=n}^{m} \frac{h_{j+1}}{h_j} \cdot h_j^{-d}||Q_j B_j^m Q_m x|| \leq$$

$$\leq \sum_{j=n}^{m} \frac{h_{j+1}}{h_j} \cdot h_j^{-d} N h_n^{\epsilon} \cdot \frac{h_{m+1}}{h_m} \left(\frac{h_j}{h_m}\right)^{\nu} ||A_m^j Q_j B_n^m Q_m x|| =$$

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$$= NH_1 h_m^{-\nu+\epsilon}||Q_m x|| \sum_{j=n}^{m} \frac{h_{j+1}}{h_j} h_j^{-d+\nu} \leq$$

$$\leq NH_1^2 h_m^{-\nu+\epsilon}||Q_m x|| \sum_{j=n}^{m} h_j^{-d+\nu} \leq$$

$$\leq NH_1^2 H_3 h_m^{-\nu+\epsilon} h_m^{-d+\nu}||Q_m x|| = Dh_m^{\epsilon-d}||Q_m x|| \leq Dh_m^{\epsilon-d}||Q_m x||$$

where $D = NH_1^2 H_3 \geq 1$ and $(m, n, p, x) \in T \times X$.

Sufficiency. For $j = n$ in $(hD_2^{s'})$. □

Corollary 5.9. *The pair $(\mathcal{A}, P)$ is uniformly h-dichotomic if and only if there exist $D \geq 1, d > 0$, with:*

$(uhD_1^{s'})$ $\sum_{k=n}^{\infty} \frac{h_{k+1}}{h_k} h_k^d ||A_k^p P_p x|| \leq Dh_n^d ||A_n^p P_p x||$

$(uhD_2^{s'})$ $\sum_{j=n}^{m} \frac{h_{j+1}}{h_j} h_j^{-d} ||B_m^j Q_m x|| \leq Dh_m^{-d} ||Q_m x||$

for all $(m, n, p, x) \in T \times X$.

Proof. Results immediat from Theorem 13 for $\epsilon = 0$. □

Corollary 5.10. *Let $h \in \mathcal{H}$. The pair $(\mathcal{A}, P)$ is exponentially dichotomic if only if there are $D \geq 1, d > 0$, and $\epsilon \geq 0$ with:*

$(eD_1^{s'})$ $\sum_{k=n}^{\infty} e^{dk} ||A_k^p P_p x|| \leq e^{n(\epsilon+d)} ||A_n^p P_p x||$

$(eD_2^{s'})$ $\sum_{j=n}^{m} e^{-dj} ||B_m^j Q_m x|| \leq De^{m(\epsilon-d)} ||Q_m x||$

for all $(m, n, p, x) \in T \times X$.

Proof. It follows from Theorem 13 for $h_m = e^m$. □

Theorem 5.11. *The pair $(\mathcal{A}, P)$ is strongly h-dichotomic if and only if there exist $D \geq 1, d > 0$, $\epsilon \geq 0$ with:*

$(shD_1^{s'})$ $\sum_{k=n}^{\infty} h_k^d ||A_k^p P_p x|| \leq Dh_n^{\epsilon+d} ||A_n^p P_p x||$

$(shD_2^{s'})$ $\sum_{j=n}^{m} h_j^{-d} ||B_m^j Q_m x|| \leq Dh_m^{\epsilon-d} ||Q_m x||$

for all $(m, n, p, x) \in T \times X$.

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Proof. Necessity. Let be $d \in (0, \nu)$.
The implication $(shd_1^{s'}) \implies (shD_1^{s'})$ is immediate from Theorem 5.
$(shd_2^{s'}) \implies (shD_2^{s'})$

$$\sum_{j=n}^{m} h_j^{-d}||B_j^m Q_m x|| = \sum_{j=n}^{m} h_j^{-d}||Q_j B_j^m Q_m x|| \leq$$

$$\leq \sum_{j=n}^{m} h_j^{-d} N h_n^{\epsilon} \left(\frac{h_j}{h_m}\right)^{\nu} ||A_m^j Q_j B_n^m Q_m x|| =$$

$$= N h_m^{-\nu+\epsilon}||Q_m x|| \sum_{j=n}^{m} h_j^{-d+\nu} \leq$$

$$\leq N h_m^{-\nu+\epsilon}||Q_m x|| \sum_{j=n}^{m} h_j^{-d+\nu} \leq$$

$$\leq N H_3 h_m^{-\nu+\epsilon} h_m^{-d+\nu}||Q_m x|| \leq D h_m^{\epsilon-d}||Q_m x||$$

where $D = NH_3 \geq 1$ and $(m, n, p, x) \in T \times X$.

Sufficiency. Results immediate from $(hD_2^{s'})$ for $j = n$.

□

Corollary 5.12. *If the pair $(\mathcal{A}, P)$ is uniformly strongly h-dichotomic if and only if there exist $D \geq 1, d > 0$, with:*

$(ushD_1^{s'})$ $\sum_{k=n}^{\infty} h_k^d ||A_k^p P_p x|| \leq D h_n^d ||A_n^p P_p x||$

$(ushD_2^{s'})$ $\sum_{j=n}^{m} h_j^{-d} ||B_m^j Q_m x|| \leq D h_m^{-d} ||Q_m x||$

for all $(m, n, p, x) \in T \times X$.

Proof. Results immediat from Theorem 14 for $\epsilon = 0$. □

Corollary 5.13. *Let $h \in \mathcal{H}$. The pair $(\mathcal{A}, P)$ is strongly exponentially dichotomic if and only if there are $D \geq 1, d > 0$, and $\epsilon \geq 0$ with:*

$(seD_1^{s'})$ $\sum_{k=n}^{\infty} e^{dk} ||A_k^p P_p x|| \leq e^{n(\epsilon+d)} ||A_n^p P_p x||$

$(seD_2^{s'})$ $\sum_{j=n}^{m} e^{-dj} ||B_m^j Q_m x|| \leq D e^{m(\epsilon-d)} ||Q_m x||$

for all $(m, n, p, x) \in T \times X$.

Proof. It follows from Theorem 14 for $h_m = e^m$. □

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