SCIENTIFIC BULLETIN
OF
THE POLITEHNICA UNIVERSITY OF TIMISOARA

Vol.69(83)	MATHEMATICS–PHISYCS	2024

In memory of Professor Borislav D. CRSTICI on the Centenary of his birth.

DYNAMICS OF a FIVE-DIMENSIONAL MATHEMATICAL MODEL FOR UNEMPLOYMENT

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Abstract

The purpose of this paper is to develop and analyse a five-dimensional mathematical model of labor market slack, incorporating both unemployment and employment characterised by limited working hours.
[1]

1 Introduction

Over time, one of the significant challenges faced by a country is economic downturn driven by unemployment. The causes of this phenomenon are numerous and vary depending on the specific context of each country. Key factors contributing to the uncontrolled rise in unemployment include issues such as population growth and sluggish economic expansion.

Previous research in the field has explored various aspects of five-dimensional mathematical models to control the unemployment, including studies involving time delays [10], [11].

Distributed time delays have been widely applied in mathematical models across different disciplines, including population biology, epidemiology, and economics [1], [2], [3], [4], [5], [6], [7], [9].

In this paper, we analyze the nonlinear unemployment model without considering time delays, utilizing the stability properties of differential equations to examine its dynamics.

[1]

Keywords and phrases: *unemployment model, equilibrium points, local stability, global stability*

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2 The mathematical model

The following five variables which describe the mathematical model for to control the unemployment are considered: the number of unemployed persons $X_1(t)$, the number of immigrants $X_2(t)$, the number of temporary employed persons $X_3(t)$, the number of regularly employed persons $X_4(t)$, respectively and the number of available vacancies $X_5(t)$ at time t.

The mathematical model proposed is:

$$\begin{cases} \dot{X}_1(t) = a_1 - a_2X_1(t)X_5(t) + a_3X_4(t) - a_4X_1(t) + a_5X_3(t) - b_1X_1(t), \\ \dot{X}_2(t) = m_1 - m_2X_2(t)X_5(t) - b_2X_2(t), \\ \dot{X}_3(t) = a_4X_1(t) - a_5X_3(t) - c_1X_3(t)X_5(t) - b_3X_3(t), \\ \dot{X}_4(t) = a_2X_1(t)X_5(t) + m_2X_2(t)X_5(t) + c_1X_3(t)X_5(t) - a_3X_4(t) - b_4X_4(t), \\ \dot{X}_5(t) = c_2X_4(t) - b_5X_5(t), \end{cases} \tag{1}$$

where: a_1 represents the constant growth rate of unemployed persons entering the labor market, a_2 is the rate of hiring, a_3 describes the rate of firing, a_4 expresses the rate of move to unemployment, a_5 - the rate of move to temporary employment; b_1 is the rate of migration of unemployed persons; b_2 - the rate of return or death, b_3 describes the rate of temporary employment, b_4 represents the rate of retirement, migration or death of employed persons; b_5 - the rate of available vacancies; c_1 is the rate of upgrading from limited hours to regular employment; c_2 is the rate of vacancies creation in response to current employment conditions; m_1 represents the exogenous increase in migration and m_2 is the migrants' entry rate into regular employment.

Considering the previously mentioned factors, we present a schematic representation of the model, as depicted in Figure 1.

3 Adimensional model

In order to write the adimensional system we consider the changes of variables:

$$\begin{cases} x_1(t) = \dfrac{a_2c_2}{a_5^2}X_1\left(\dfrac{t}{a_5}\right), \\ x_2(t) = \dfrac{a_2c_2}{a_5^2}X_2\left(\dfrac{t}{a_5}\right), \\ x_3(t) = \dfrac{a_2c_2}{a_5^2}X_3\left(\dfrac{t}{a_5}\right), \\ x_4(t) = \dfrac{a_2c_2}{a_5^2}X_4\left(\dfrac{t}{a_5}\right), \\ x_5(t) = \dfrac{a_2}{a_5}X_5\left(\dfrac{t}{a_5}\right), \end{cases} \tag{2}$$

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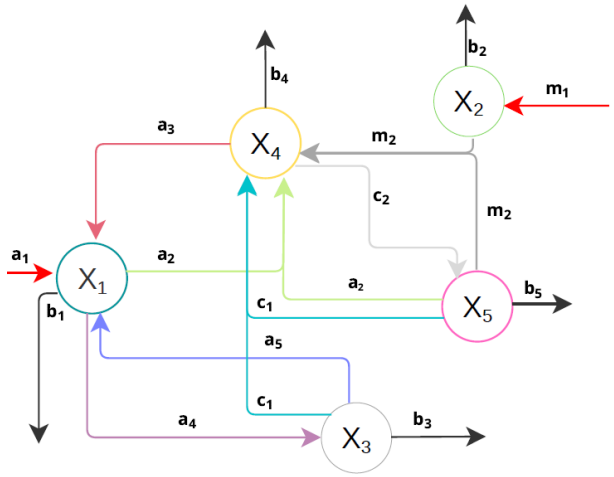


Figure 1: Representation of the five-dimensional model without time delays given in (1).

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and the system (1) becomes:

$$\begin{cases} \dot{x}_1(t) = \gamma_1 - x_1(t)x_5(t) + \alpha_3 x_4(t) - \alpha_4 x_1(t) + x_3(t) - \beta_1 x_1(t), \\ \dot{x}_2(t) = \gamma_2 - \alpha_2 x_2(t)x_5(t) - \beta_2 x_2(t), \\ \dot{x}_3(t) = \alpha_4 x_1(t) - x_3(t) - \alpha_1 x_3(t)x_5(t) - \beta_3 x_3(t), \\ \dot{x}_4(t) = x_1(t)x_5(t) + \alpha_2 x_2(t)x_5(t) + \alpha_1 x_3(t)x_5(t) - \alpha_3 x_4(t) - \beta_4 x_4(t), \\ \dot{x}_5(t) = x_4(t) - \beta_5 x_5(t), \end{cases} \tag{3}$$

where the coefficients are expressed as:

$$\gamma_1 = \frac{a_1 a_2 c_2}{a_5^3}, \quad \gamma_2 = \frac{a_2 c_2 m_1}{a_5^3},$$
$$\alpha_1 = \frac{c_1}{a_2}, \quad \alpha_2 = \frac{m_2}{a_2}, \quad \alpha_3 = \frac{a_3}{a_5}, \quad \alpha_4 = \frac{a_4}{a_5},$$
$$\beta_1 = \frac{b_1}{a_5}, \quad \beta_2 = \frac{b_2}{a_5}, \quad \beta_3 = \frac{b_3}{a_5}, \quad \beta_4 = \frac{b_4}{a_5}, \quad \beta_5 = \frac{b_5}{a_5},$$

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4 Equilibrium points of the model

As constant solutions of system (3), the equilibrium points satisfy the following algebraic equations:

$$\begin{cases} \gamma_1 - x_1x_5 + \alpha_3x_4 - \alpha_4x_1 + x_3 - \beta_1x_1 = 0, \\ \gamma_2 - \alpha_2x_2x_5 - \beta_2x_2 = 0, \\ \alpha_4x_1 - x_3 - \alpha_1x_3x_5 - \beta_3x_3 = 0, \\ x_1x_5 + \alpha_2x_2x_5 + \alpha_1x_3x_5 - \alpha_3x_4 - \beta_4x_4 = 0, \\ x_4 - \beta_5x_5 = 0. \end{cases} \tag{4}$$

On the one hand, we note that system (4) has at least one solution, which corresponds to the case $x_5 = 0$. As a result, we obtain the equilibrium point S^0:

$$S^0 := (\delta_1(1+\beta_3), \delta_2, \delta_1\alpha_4, 0, 0)\,,$$

with

$$\delta_1 = \frac{\gamma_1}{\beta_1 + \beta_1\beta_3 + \alpha_4\beta_3} \quad \text{and} \quad \delta_2 = \frac{\gamma_2}{\beta_2}.$$

This equilibrium point represents the situation in which there is neither regular employment and no available vacancies.

Now we introduce the basic reproduction number R_0. It serves as a threshold parameter that predicts whether or not problems with immigration, unemployment, and temporary employment will increase or decrease. We determine R_0 using the next-generation matrix method:

$$R_0 = \frac{\delta_1(1+\beta_3+\alpha_1\alpha_4) + \delta_2\alpha_2}{(\alpha_3+\beta_4)\beta_5}.$$

On the other hand, if and only if x_5 is the solution of the following cubic equation, system (4) has at least one solution with the last component $x_5 > 0$:

$$E_3x_5^3 + E_2x_5^2 + E_1x_5 + E_0 = 0, \tag{5}$$

with the coefficients

$$\begin{aligned} E_3 &= \alpha_1\beta_4\beta_5, \\ E_2 &= \alpha_1(\nu_2\beta_4\beta_5 - \gamma_1 - \gamma_2) + \beta_4\beta_5(1+\beta_3+\alpha_1\alpha_4) + \alpha_1\beta_1\mu, \\ E_1 &= (\nu_2\beta_4\beta_5 - \gamma_1 - \gamma_2)(1+\beta_3+\alpha_1\alpha_4) - \nu_2\gamma_1\alpha_1 + \mu\beta_1\Big(1+\beta_3+\alpha_1\alpha_4\frac{\nu_1}{\beta_1}\Big) \\ &\quad + (\mu\nu_2 - \gamma_2)\beta_1\alpha_1, \\ E_0 &= \frac{\mu\gamma_1\gamma_2}{\alpha_2\delta_1\delta_2}(1-R_0), \end{aligned}$$

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where R_0, δ_1 and δ_2 are given above and

$$\nu_1 = \frac{\beta_3}{\alpha_1}, \qquad \nu_2 = \frac{\beta_2}{\alpha_2}, \qquad \mu = (\alpha_3 + \beta_4)\beta_5.$$

System (3) in this case has at least one equilibrium point S^+ with the form:

$$S^+ := \left(-\frac{Q(x_5) + \nu_1 d(x_5)}{(\beta_1 - \nu_1)(x_5 + \nu_2)}, \frac{\gamma_2}{\alpha_2(\nu_2 + x_5)}, \frac{Q(x_5) + \beta_1 d(x_5)}{\alpha_1(\beta_1 - \nu_1)(\nu_2 + x_5)}, \beta_5 x_5, x_5 \right),$$

where

$$\begin{aligned} Q(x_5) &= (\beta_4\beta_5 x_5 - \gamma_1 - \gamma_2)(x_5 + \nu_2) + \gamma_2\nu_2, \\ d(x_5) &= \mu(x_5 + \nu_2) - \gamma_2. \end{aligned}$$

We actually distinguish between two cases:

Case 1: If $R_0 > 1$ then $E_0 < 0$ and system (3) has at least one positive equilibrum point. Moreover, if either $E_1 < 0$ or $E_1 > 0$ and $E_2 > 0$, Descartes' rule of signs guarantees the existence of a unique positive equilibrium point S^+.

Case 2: If $R_0 < 1$ then $E_0 > 0$, we may have either two or zero positive equilibrium points. For instance, if $E_1 > 0$ and $E_2 > 0$, system (3) does not have positive equilibrium points.

5 Positivity and boundedness of solutions

Theorem 5.1. *The open positive (nonnegative) orthant $\mathbb{R}^5_+ = (0, \infty)^5$ is invariant to the flow of system* (3)*. Furthermore, denoting $\beta_m = \min(\beta_1, \beta_2, \beta_3, \beta_4)$, the set*

$$\Omega = \left\{ (x_1, x_2, x_3, x_4, x_5) : 0 \leq x_1 + x_2 + x_3 + x_4 \leq \frac{\gamma_1 + \gamma_2}{\beta_m},\ 0 \leq x_5 \leq \frac{\gamma_1 + \gamma_2}{\beta_m \beta_5} \right\}$$

is a region of attraction for the system (1)*, attracting all the solutions with initial conditions in the open positive orthant $\mathbb{R}^5_+$.*

Proof. The positivity of solutions is proved using Theorem 3.4. from [12].

For the boundedness of solutions, from the first four equations of system (3) we observe that

$$\frac{d}{dt}[x_1(t) + x_2(t) + x_3(t) + x_4(t)] = \gamma_1 + \gamma_2 - \beta_1 x_1(t) - \beta_2 x_2(t) - \beta_3 x_3(t) - \beta_4 x_4(t),$$

and taking into account that $\beta_m = \min(\beta_1, \beta_2, \beta_3, \beta_4)$ we have

$$\frac{d}{dt}[x_1(t)+x_2(t)+x_3(t)+x_4(t)] \leq \gamma_1+\gamma_2-\beta_m[x_1(t)+x_2(t)+x_3(t)+x_4(t)], \quad \forall\, t > 0.$$

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Therefore, using basic differential inequality techniques, we obtain:

$$x_1(t)+x_2(t)+x_3(t)+x_4(t)\leq e^{-\beta_m t}\left(x_1(0)+x_2(0)+x_3(0)+x_4(0)-\frac{\gamma_1+\gamma_2}{\beta_m}\right)+\frac{\gamma_1+\gamma_2}{\beta_m},$$

for any $t \geq 0$.

On one hand, if $x_1(0)+x_2(0)+x_3(0)+x_4(0) \leq \frac{\gamma_1+\gamma_2}{\beta_m}$ it follows that $(x_1(0)+x_2(0)+x_3(0)+x_4(0) \leq \frac{\gamma_1+\gamma_2}{\beta_m}$, for any $t > 0$. Otherwise, if $x_1(0)+x_2(0)+x_3(0)+x_4(0) > \frac{\gamma_1+\gamma_2}{\beta_m}$, we have

$$\limsup_{t\to\infty}[x_1(t)+x_2(t)+x_3(t)+x_4(t)] \leq \frac{\gamma_1+\gamma_2}{\beta_m}.$$

Moreover, the last equation of (3) gives

$$\frac{d}{dt}[x_5(t)] = x_4(t)-\beta_5 x_5(t) \leq \frac{\gamma_1+\gamma_2}{\beta_m}-\beta_5 x_5(t).$$

Applying the generalized l'Hospital rule, we deduce:

$$\limsup_{t\to\infty} x_5(t) \leq \frac{\gamma_1+\gamma_2}{\beta_m\beta_5},$$

and the desired conclusion is proved.

□

Remark 5.2. *The previous theorem states that solutions of system* (3) *originating from initial conditions belonging to the open positive orthant of* $\mathbb{R}^5$, *i.e.* $\varphi_i : (-\infty, 0] \to (0, \infty)$, *for* $i = \overline{1,5}$, *remain positive for any* $t > 0$. *Moreover, such solutions satisfy the inequalities:*

$$\limsup_{t\to\infty}[x_1(t)+x_2(t)+x_3(t)+x_4(t)] \leq \frac{\gamma_1+\gamma_2}{\beta_m} \quad \textit{and} \quad \limsup_{t\to\infty} x_5(t) \leq \frac{\gamma_1+\gamma_2}{\beta_5\beta_m}.$$

5.1 Local stability analysis for S^0

Theorem 5.3. *The equilibrium point* S^0 *of system (3) is locally asymptotically stable if and only if* $R_0 < 1$.

Proof. The characteristic equation is:

$(\lambda+\beta_2)[\lambda^2+\lambda(\beta_3+\beta_1+\alpha_4+1)+\alpha_4\beta_3+\beta_1\beta_3+\beta_1][\lambda^2+\lambda(\alpha_3+\beta_3+\beta_5)$ $+\beta_5(\alpha_3+\beta_3)(1-R_0)] = 0$.

We observe that $\lambda = -\beta_2 < 0$ is a root of this characteristic equation.

The first quadratic polynomial has all the roots with negative real part because all the coefficients are positive.

For the second quadratic polynomial, based on the Routh-Hurwitz conditions, we obtain that all the roots have negative real part if and only if $R_0 < 1$. □

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5.2 Global stability analysis for S^0

Theorem 5.4. *The equilibrium point S^0 is globally asymptotically stable in the open positive orthant $\mathbb{R}^5_+$, if the following inequality holds:*

$$R_0 < \frac{\beta_4}{\alpha_3 + \beta_4}. \tag{6}$$

Proof. Considering an arbitrary solution $x_i(t)$, $i = \overline{1,5}$, of system (3), with initial conditions $\varphi_i : (-\infty, 0] \to (0, \infty)$. Using Theorem 5.1 it follows that $x_i(t) > 0$, for any $t > 0$ and $i = \overline{1,5}$. Consequently, we consider the functions:

$$L_k(t) = x_k(t) - x_k^0 - x_k^0 \ln \frac{x_k(t)}{x_k^0} \ , \ k = \overline{1,5} \ ,$$

and we denote:

$$E_k(t) = 1 - \frac{x_k(t)}{x_k^0} + \ln \frac{x_k(t)}{x_k^0} < 0, \qquad \tilde{E_k}(t) = 1 - \frac{x_k^0}{x_k(t)} + \ln \frac{x_k^0}{x_k(t)} < 0 \ , \ k = \overline{1,5}.$$

Now, we have:

$$\begin{aligned}
L_1'(t) &= \dot{x}_1\Big(1 - \frac{x_1^0}{x_1}\Big) = \Big(1 - \frac{x_1^0}{x_1}\Big)[\gamma_1 - x_1x_5 + \alpha_3x_4 + x_3 - (\alpha_4 + \beta_1)x_1] \\
&= \Big(1 - \frac{x_1^0}{x_1}\Big)[x_1^0x_5^0 - \alpha_3x_4^0 - x_3^0 + (\alpha_4 + \beta_1)x_1^0 - x_1x_5 + \alpha_3x_4 + x_3 - (\alpha_4 + \beta_1)x_1] \\
&= \Big(1 - \frac{x_1^0}{x_1}\Big)\Big[(\alpha_4 + \beta_1)x_1^0\Big(1 - \frac{x_1}{x_1^0}\Big) - x_3^0\Big(1 - \frac{x_3}{x_3^0}\Big) - x_1x_5 + \alpha_3x_4\Big] \\
&= (\alpha_4 + \beta_1)x_1^0\left(2 - \frac{x_1^0}{x_1} - \frac{x_1}{x_1^0}\right) - x_3^0\left(1 + \frac{x_1^0}{x_1} \cdot \frac{x_3}{x_3^0} - \frac{x_3}{x_3^0} - \frac{x_1^0}{x_1}\right) \\
&- x_1x_5 + \alpha_3x_4 - \alpha_3x_4\frac{x_1^0}{x_1} + x_5x_1^0 \\
&\leq (\alpha_4 + \beta_1)x_1^0\left(2 - \frac{x_1^0}{x_1} - \frac{x_1}{x_1^0}\right) - x_3^0\left(2 + \ln\frac{x_1^0}{x_1} + \ln\frac{x_3}{x_3^0} - \frac{x_3}{x_3^0} - \frac{x_1^0}{x_1}\right) \\
&- x_1x_5 + \alpha_3x_4 + x_5x_1^0 \\
&= (\alpha_4 + \beta_1)x_1^0(E_1 + \tilde{E_1}) - x_3^0(\tilde{E_1} + E_3) - x_1x_5 + x_1^0x_5 + \alpha_3x_4 - \alpha_3x_4\frac{x_1^0}{x_1}.
\end{aligned}$$

In addition, we compute:

$$L_2'(t) = \dot{x}_2\left(1 - \frac{x_2^0}{x_2}\right) = \left(1 - \frac{x_2^0}{x_2}\right)(\gamma_2 - \alpha_2x_2x_5 - \beta_2x_2)$$

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$$= \left(1 - \frac{x_2^0}{x_2}\right)(\alpha_2 x_2^0 x_5^0 + \beta_2 x_2^0 - \alpha_2 x_2 x_5 - \beta_2 x_2)$$
$$= \beta_2 x_2^0 \left(2 - \frac{x_2^0}{x_2} - \frac{x_2}{x_2^0}\right) - \alpha_2 x_2 x_5 + \alpha_2 x_2^0 x_5$$
$$= \beta_2 x_2^0 (E_2 + \tilde{E}_2) - \alpha_2 x_2 x_5 + \alpha_2 x_2^0 x_5,$$

$$L_3'(t) = \dot{x}_3 \left(1 - \frac{x_3^0}{x_3}\right) = \left(1 - \frac{x_3^0}{x_3}\right)(\alpha_4 x_1 - x_3 - \alpha_1 x_3 x_5 - \beta_3 x_3)$$
$$= \left(1 - \frac{x_3^0}{x_3}\right)\left[\alpha_4 (x_1 - x_1^0) - (1 + \beta_3)(x_3 - x_3^0) - \alpha_1 (x_3 x_5 - x_3^0 x_5^0)\right]$$
$$= \left(1 - \frac{x_3^0}{x_3}\right)\left[\alpha_4 x_1^0 \left(\frac{x_1}{x_1^0} - 1\right) - x_3^0 (1 + \beta_3)\left(\frac{x_3}{x_3^0} - 1\right) - \alpha_1 x_3 x_5\right]$$
$$= -\alpha_4 x_1^0 \left(1 + \frac{x_3^0}{x_3} \cdot \frac{x_1}{x_1^0} - \frac{x_3^0}{x_3} - \frac{x_1}{x_1^0}\right) + x_3^0 (1 + \beta_3)\left(2 - \frac{x_3}{x_3^0} - \frac{x_3^0}{x_3}\right)$$
$$- \alpha_1 x_3 x_5 + \alpha_1 x_3^0 x_5$$
$$\leq -\alpha_4 x_1^0 \left(2 + \ln \frac{x_3^0}{x_3} + \ln \frac{x_1}{x_1^0} - \frac{x_3^0}{x_3} - \frac{x_1}{x_1^0}\right) + x_3^0 (1 + \beta_3)\left(2 - \frac{x_3}{x_3^0} - \frac{x_3^0}{x_3}\right)$$
$$- \alpha_1 x_3 x_5 + \alpha_1 x_3^0 x_5$$
$$= -\alpha_4 x_1^0 (E_1 + \tilde{E}_3) + (1 + \beta_3) x_3^0 (E_3 + \tilde{E}_3) - \alpha_1 x_3 x_5 + \alpha_1 x_3^0 x_5.$$

Moreover, considering

$$L_{4,5}(t) = x_4 + c x_5,$$

we have

$$L_{4,5}'(t) = \dot{x}_4 + c\dot{x}_5$$
$$= x_1 x_5 + \alpha_2 x_2 x_5 + \alpha_1 x_3 x_5 - (\alpha_3 + \beta_4 - c) x_4 - c\beta_5 x_5.$$

Further, considering

$$L(t) = L_1(t) + L_2(t) + L_3(t) + L_{4,5}(t)$$

we compute

$$L'(t) \leq (\alpha_4 + \beta_1) x_1^0 (E_1 + \tilde{E}_1) - x_3^0 (\tilde{E}_1 + E_3) - x_1 x_5 + x_1^0 x_5 + \alpha_3 x_4$$
$$+ \beta_2 x_2^0 (E_2 + \tilde{E}_2) - \alpha_2 x_2 x_5 + \alpha_2 x_2^0 x_5$$
$$- \alpha_4 x_1^0 (E_1 + \tilde{E}_3) + (1 + \beta_3) x_3^0 (E_3 + \tilde{E}_3) - \alpha_1 x_3 x_5 + \alpha_1 x_3^0 x_5$$
$$+ x_1 x_5 + \alpha_2 x_2 x_5 + \alpha_1 x_3 x_5 - (\alpha_3 + \beta_4 - c) x_4 - c\beta_5 x_5$$
$$= \beta_1 x_1^0 E_1 + \left[(\alpha_4 + \beta_1) x_1^0 - x_3^0\right] \tilde{E}_1 + \beta_2 x_2^0 (E_2 + \tilde{E}_2) + \beta_3 x_3^0 E_3$$

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$$+\left[(1+\beta_3)x_3^0-\alpha_4x_1^0\right]\tilde{E}_3-(\beta_4-c)x_4-(c\beta_5-x_1^0-\alpha_2x_2^0-\alpha_1x_3^0)x_5$$
$$=\beta_1x_1^0E_1+\beta_2x_2^0(E_2+\tilde{E}_2)+\beta_3x_3^0E_3-(\beta_4-c)x_4-\beta_5\left[c-R_0(\alpha_3+\beta_4)\right]x_5.$$

Due to the fact that the inequality $R_0<\frac{\beta_4}{\alpha_3+\beta_4}$ is true, there is $c>0$ such that $L'(t)<0$. We conclude that the equilibrium point S^0 of the system (3) is globally asymptotically stable in $\mathbb{R}^5_+$ using LaSalle's invariance principle [8], [12]. □

5.3 Local stability analysis for S^+

The characteristic equation is:

$$P_0(\lambda)+P_1(\lambda)+P_2(\lambda)+P_{12}(\lambda)=0 \tag{7}$$

where

$$\begin{aligned}
P_0(\lambda)&=-(\beta_5+\lambda)(\beta_2+\alpha_2x_5+\lambda)\{(\beta_4+\lambda)[x_5+\lambda+(\alpha_4+x_5+\lambda)\\
&\quad(\beta_3+\alpha_1x_5+\lambda)]+\alpha_3[\alpha_4(\beta_3+\lambda)+\lambda(1+\beta_3+\alpha_1x_5+\lambda)]\}\\
P_1(\lambda)&=-\beta_1(\alpha_3+\beta_4+\lambda)(\beta_5+\lambda)(1+\beta_3+\alpha_1x_5+\lambda)(\beta_2+\alpha_2x_5+\lambda)\\
P_2(\lambda)&=\alpha_2^2\alpha_4x_2x_5(\beta_3+\alpha_1x_5+\lambda)-\alpha_2^2x_2x_5(1+\beta_3+\alpha_1x_5+\lambda)(\lambda+x_5)\\
&\quad-\alpha_4\beta_5(\alpha_3+\beta_4)(\beta_2+\alpha_2x_5+\lambda)-\alpha_1\alpha_4(\beta_2+\alpha_2x_5+\lambda)(x_1x_5+\alpha_1x_3x_5)\\
&\quad+\beta_5(\alpha_3+\beta_4)(1+\beta_3+\alpha_1x_5+\lambda)(\beta_2+\alpha_2x_5+\lambda)(\alpha_4+x_5+\lambda)\\
&\quad-x_5(\beta_2+\alpha_2x_5+\lambda)[\alpha_1x_3+x_1(1+\beta_3+\alpha_1x_5+\lambda)]\\
&\quad-\alpha_1^2x_3x_5(\beta_2+\alpha_2x_5+\lambda)(x_5+\lambda)\\
P_{12}(\lambda)&=\beta_1[-\alpha_2^2x_2x_5(1+\beta_3+\alpha_1x_5+\lambda)-\alpha_1^2x_3x_5(\beta_2+\alpha_2x_5+\lambda)\\
&\quad+\beta_5(\alpha_3+\beta_4)(1+\beta_3+\alpha_1x_5+\lambda)(\beta_2+\alpha_2x_5+\lambda)]
\end{aligned}$$

The algebraic equation of fifth degree for the positive equilibrium point is:

$$\lambda^5+a_{45}\lambda^4+a_{35}\lambda^3+a_{25}\lambda^2+a_{15}\lambda+a_{05}=0, \tag{8}$$

where the coefficients are:

$$\begin{aligned}
a_{45}&=1+\alpha_3+\alpha_4+\beta_1+\beta_2+\beta_3+\beta_4+\beta_5+(1+\alpha_1+\alpha_2)x_5\\
a_{35}&=\beta_2+\alpha_4\beta_2+\alpha_4\beta_3+\beta_2\beta_3+\beta_4+\alpha_4\beta_4+\beta_2\beta_4+\beta_3\beta_4+\beta_5+\alpha_4\beta_5+\beta_2\beta_5\\
&\quad+\beta_3\beta_5+[1+\beta_2+\beta_3+\beta_4+\beta_5+\alpha_1(\alpha_4+\beta_2+\beta_4+\beta_5)\\
&\quad+\alpha_2(1+\alpha_4+\beta_3+\beta_4+\beta_5)]x_5+(\alpha_1+\alpha_2+\alpha_1\alpha_2)x_5^2\\
&\quad+\alpha_3[1+\alpha_4+\beta_1+\beta_2+\beta_3+(\alpha_1+\alpha_2)x_5]\\
&\quad+\beta_1[1+\beta_2+\beta_3+\beta_4+\beta_5+(\alpha_1+\alpha_2)x_5]\\
a_{25}&=\alpha_3\beta_1+\alpha_3\beta_2+\alpha_3\alpha_4\beta_2+\beta_1\beta_2+\alpha_3\beta_1\beta_2+\alpha_3\alpha_4\beta_3+\alpha_3\beta_1\beta_3+\alpha_3\beta_2\beta_3\\
&\quad+\alpha_4\beta_2\beta_3+\beta_1\beta_2\beta_3+\beta_1\beta_4+\beta_2\beta_4+\alpha_4\beta_2\beta_4+\beta_1\beta_2\beta_4+\alpha_4\beta_3\beta_4+\beta_1\beta_3\beta_4
\end{aligned}$$

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$$
\begin{aligned}
&+\beta_2\beta_3\beta_4+\beta_1\beta_5+\beta_2\beta_5+\alpha_4\beta_2\beta_5+\beta_1\beta_2\beta_5+\alpha_4\beta_3\beta_5+\beta_1\beta_3\beta_5+\beta_2\beta_3\beta_5\\
&+\{\beta_2+\beta_2\beta_3+\beta_4+\beta_2\beta_4+\beta_3\beta_4+\beta_5+(-\alpha_3+\beta_2+\beta_3)\beta_5+\alpha_1[\beta_1\beta_2\\
&+\alpha_3(\beta_1+\beta_2)+\beta_1\beta_4+\beta_2\beta_4+(\beta_1+\beta_2)\beta_5+\alpha_4(\beta_2+\beta_4+\beta_5)]\\
&+\alpha_2[\alpha_3(1+\alpha_4+\beta_1+\beta_3)+\beta_4+\beta_5+\beta_3(\beta_4+\beta_5)+\alpha_4(\beta_3+\beta_4+\beta_5)\\
&+\beta_1(1+\beta_3+\beta_4+\beta_5)]+x_1+\alpha_2^2x_2+\alpha_1^2x_3\}x_5+\{\alpha_1(\beta_2+\beta_4+\beta_5)\\
&+\alpha_2[1+\beta_3+\beta_4+\beta_5+\alpha_1(\alpha_3+\alpha_4+\beta_1+\beta_4+\beta_5)]\}x_5^2+\alpha_1\alpha_2x_5^3
\end{aligned}
$$

$$
\begin{aligned}
a_{15}&=\beta_2[\beta_1+(\alpha_4+\beta_1)\beta_3](\alpha_3+\beta_4+\beta_5)+\{\beta_2\beta_4+\beta_2\beta_3\beta_4-\alpha_3\beta_5+\beta_2\beta_5\\
&-\alpha_3\beta_2\beta_5-\alpha_3\beta_3\beta_5+\beta_2\beta_3\beta_5+\alpha_2[\beta_1+(\alpha_4+\beta_1)\beta_3](\alpha_3+\beta_4+\beta_5)\\
&+(1+\beta_2+\beta_3)x_1+\alpha_2^2(1+\alpha_4+\beta_1+\beta_3)x_2+\alpha_1^2(\alpha_4+\beta_1+\beta_2)x_3\\
&+\alpha_1[(\alpha_4+\beta_1)\beta_2(\beta_4+\beta_5)+\alpha_3(\beta_1\beta_2-\alpha_4\beta_5)+\alpha_4x_1+x_3]\}x_5\\
&+\{\alpha_2[\beta_4+\beta_3\beta_4+\beta_5-\alpha_3\beta_5+\beta_3\beta_5+x_1+\alpha_2x_2]+\alpha_1\{-\alpha_3\beta_5+\beta_2(\beta_4+\beta_5)\\
&+\alpha_2[\alpha_3\beta_1+(\alpha_4+\beta_1)(\beta_4+\beta_5)]+x_1+\alpha_2^2x_2\}+\alpha_1^2(1+\alpha_2)x_3\}x_5^2\\
&+\alpha_1\alpha_2(\beta_4+\beta_5)x_5^3\\
a_{05}&=x_5\{\alpha_2^2\beta_1x_2-\alpha_3\beta_5[\beta_2+\alpha_2x_5][1+\beta_3+\alpha_1(\alpha_4+x_5)]+\beta_2\{x_1[1+\beta_3\\
&+\alpha_1(\alpha_4+x_5)]+\alpha_1x_3[1+\alpha_1(\alpha_4+\beta_1+x_5)]\}+\alpha_2\{\alpha_2x_2[x_5+(\alpha_4+\beta_1+x_5)\\
&(\beta_3+\alpha_1x_5)]+x_5\{x_1[1+\beta_3+\alpha_1(\alpha_4+x_5)]+\alpha_1x_3[1+\alpha_1(\alpha_4+\beta_1+x_5)]\}\}\}
\end{aligned}
$$

We write the Routh Hurwitz matrix as:

$$
\begin{pmatrix}
a_{45} & a_{25} & a_{05} & 0 & 0\\
1 & a_{35} & a_{14} & 0 & 0\\
0 & a_{45} & a_{25} & a_{05} & 0\\
0 & 1 & a_{35} & a_{15} & 0\\
0 & 0 & a_{45} & a_{25} & a_{05}
\end{pmatrix}
$$

The minors of the Routh Hurwitz matrix are given by the formulas:

$$\Delta_1=a_{45}>0,$$

$$\Delta_2=\begin{vmatrix} a_{45} & a_{25}\\ 1 & a_{35}\end{vmatrix}=a_{35}a_{45}-a_{25}>0,$$

$$\Delta_3=\begin{vmatrix} a_{45} & a_{25} & a_{05}\\ 1 & a_{35} & a_{15}\\ 0 & a_{45} & a_{25}\end{vmatrix}=a_{25}a_{35}a_{45}+a_{05}a_{44}-R_1R_4^2-R_2^2=R_2\Delta_2,$$

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$$\Delta_4 = \begin{vmatrix} R_4 & R_2 & R_0 & 0 \\ 1 & R_3 & R_1 & 0 \\ 0 & R_4 & R_2 & R_0 \\ 0 & 1 & R_3 & R_1 \end{vmatrix} =$$

$$=\mathrm{R}_1R_2R_3R_4 + 2R_0R_1R_4 - R_0R_3^2R_4 - R_1^2R_4^2 - R_1R_2^2 - R_0^2 + R_0R_2R_3,$$

$$\Delta_5 = \begin{vmatrix} R_4 & R_2 & R_0 & 0 & 0 \\ 1 & R_3 & R_1 & 0 & 0 \\ 0 & R_4 & R_2 & R_0 & 0 \\ 0 & 1 & R_3 & R_1 & 0 \\ 0 & 0 & R_4 & R_2 & R_0 \end{vmatrix} = R_0\Delta_4.$$

Considering the complexity of the equation, we conduct a numerical analysis of the roots.

5.4 Global stability analysis for S^+

Theorem 5.5. *The positive equilibrium point S^+ is globally asymptotically stable in the open positive orthant $\mathbb{R}^5_+$ if $a_3 = 0$.*

Proof. Let us consider an arbitrary solution $x_i(t)$, $i = \overline{1,5}$, of system (3), with initial conditions $\varphi_i : (-\infty, 0] \to (0, \infty)$. From Theorem 5.1 it follows that $x_i(t) > 0$, for any $t > 0$ and $i = \overline{1,5}$.

Knowing that $x - 1 - \ln x \geq 0$, for any $x > 0$, the Lyapunov function is constructed as follows.:

$$L(t) = L_1(t) + L_2(t) + L_3(t) + L_4(t) + L_5(t), \tag{9}$$

where

$$L_k(t) = x_k(t) - x_k^+ - x_k^+ \ln \frac{x_k(t)}{x_k^+} \ , \ k = \overline{1,5} \ , \tag{10}$$

and we denote:

$$E_k(t) = 1 - \frac{x_k(t)}{x_k^+} + \ln \frac{x_k(t)}{x_k^+} \leq 0 \quad , \quad \tilde{E_k}(t) = 1 - \frac{x_k^+}{x_k(t)} + \ln \frac{x_k^+}{x_k(t)} \leq 0 \quad , \quad k = \overline{1,5}.$$

With the notations previously described and considering the fact that S^+ is an equilibrium point of system with (3), we have:

$$L_1'(t) = \dot{x}_1 \left(1 - \frac{x_1^+}{x_1}\right) = \left(1 - \frac{x_1^+}{x_1}\right) [\gamma_1 - x_1x_5 + \alpha_3x_4 + x_3 - (\alpha_4 + \beta_1)x_1]$$

$$= \left(1 - \frac{x_1^+}{x_1}\right) [x_1^+x_5^+ - \alpha_3x_4^+ - x_3^+ + (\alpha_4 + \beta_1)x_1^+ - x_1x_5 + \alpha_3x_4 + x_3$$

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$$
\begin{aligned}
&- (\alpha_4 + \beta_1)x_1] \\
&= \left(1 - \frac{x_1^+}{x_1}\right)\left[x_1^+ x_5^+ \left(1 - \frac{x_1}{x_1^+}\frac{x_5}{x_5^+}\right) - \alpha_3 x_4^+ \left(1 - \frac{x_4}{x_4^+}\right) - x_3^+ \left(1 - \frac{x_3}{x_3^+}\right)\right. \\
&\left. + (\alpha_4 + \beta_1)x_1^+ \left(1 - \frac{x_1}{x_1^+}\right)\right] \\
&= x_1^+ x_5^+ \left(1 - \frac{x_1}{x_1^+}\frac{x_5}{x_5^+} - \frac{x_1^+}{x_1} + \frac{x_5}{x_5^+}\right) - \alpha_3 x_4^+ \left(1 - \frac{x_4}{x_4^+} - \frac{x_1^+}{x_1} + \frac{x_1^+}{x_1}\frac{x_4}{x_4^+}\right) \\
&- x_3^+ \left(1 - \frac{x_3}{x_3^+} - \frac{x_1^+}{x_1} + \frac{x_1^+}{x_1}\frac{x_3}{x_3^+}\right) + (\alpha_4 + \beta_1)x_1^+ \left(2 - \frac{x_1}{x_1^+} - \frac{x_1^+}{x_1}\right) \\
&\leq x_1^+ x_5^+ \left(1 - \frac{x_1}{x_1^+}\frac{x_5}{x_5^+} - \frac{x_1^+}{x_1} + \frac{x_5}{x_5^+}\right) - \alpha_3 x_4^+ \left(2 + \ln\frac{x_1^+}{x_1} + \ln\frac{x_4}{x_4^+} - \frac{x_4}{x_4^+}\right. \\
&\left. - \frac{x_1^+}{x_1} - x_3 \left(2 + \ln\frac{x_1^+}{x_1} + \ln\frac{x_3}{x_3^+} - \frac{x_1^+}{x_1} - \frac{x_3}{x_3^+}\right) + (\alpha_4 + \beta_1)x_1^+ \left(E_1 + \tilde{E}_1\right)\right. \\
&= x_1^+ x_5^+ \left(1 - \frac{x_1}{x_1^+}\frac{x_5}{x_5^+} - \frac{x_1^+}{x_1} + \frac{x_5}{x_5^+}\right) - \alpha_3 x_4^+ \left(\tilde{E}_1 + E_4\right) - x_3^+ \left(\tilde{E}_1 + E_3\right) \\
&+ (\alpha_4 + \beta_1)x_1^+ \left(E_1 + \tilde{E}_1\right).
\end{aligned}
$$

Additionally, we obtain:

$$
\begin{aligned}
L_2'(t) = \dot{x}_2 \left(1 - \frac{x_2^+}{x_2}\right) &= \left(1 - \frac{x_2^+}{x_2}\right)(\gamma_2 - \alpha_2 x_2 x_5 - \beta_2 x_2) \\
&= \left(1 - \frac{x_2^+}{x_2}\right)(\alpha_2 x_2^+ x_5^+ + \beta_2 x_2^+ - \alpha_2 x_2 x_5 - \beta_2 x_2) \\
&= \left(1 - \frac{x_2^+}{x_2}\right)\left[\alpha_2 x_2^+ x_5^+ \left(1 - \frac{x_2}{x_2^+}\frac{x_5}{x_5^+}\right) + \beta_2 x_2^+ \left(1 - \frac{x_2}{x_2^+}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_2 x_2^+ x_5^+ \left(1 - \frac{x_2}{x_2^+}\frac{x_5}{x_5^+} - \frac{x_2^+}{x_2} + \frac{x_5}{x_5^+}\right) + \beta_2 x_2^+ \left(2 - \frac{x_2^+}{x_2} - \frac{x_2}{x_2^+}\right) \\
&= \alpha_2 x_2^+ x_5^+ \left(1 - \frac{x_2}{x_2^+}\frac{x_5}{x_5^+} - \frac{x_2^+}{x_2} + \frac{x_5}{x_5^+}\right) + \beta_2 x_2^+ \left(E_2 + \tilde{E}_2\right).
\end{aligned}
$$

Moreover, we get:

$$
\begin{aligned}
L_3'(t) = \dot{x}_3 \left(1 - \frac{x_3^+}{x_3}\right) &= \left(1 - \frac{x_3^+}{x_3}\right)(\alpha_4 x_1 - x_3 - \alpha_1 x_3 x_5 - \beta_3 x_3) \\
&= \left(1 - \frac{x_3^+}{x_3}\right)\left[\alpha_4(x_1 - x_1^+) - (1 + \beta_3)(x_3 - x_3^+) - \alpha_1(x_3 x_5 - x_3^+ x_5^+)\right]
\end{aligned}
$$

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$$
\begin{aligned}
&= \left(1-\frac{x_3^+}{x_3}\right)\left[\alpha_4 x_1^+\left(\frac{x_1}{x_1^+}-1\right)-x_3^+(1+\beta_3)\left(\frac{x_3}{x_3^+}-1\right)-\alpha_1 x_3^+ x_5^+\right.\\
&\left.\left(\frac{x_3}{x_3^+}\frac{x_5}{x_5^+}-1\right)\right] = \alpha_4 x_1^+\left(\frac{x_1}{x_1^+}-1-\frac{x_3^+}{x_3}\cdot\frac{x_1}{x_1^+}+\frac{x_3^+}{x_3}\right)+x_3^+(1+\beta_3)\\
&\left(2-\frac{x_3}{x_3^+}-\frac{x_3^+}{x_3}\right)+\alpha_1 x_3^+ x_5^+\left(1-\frac{x_3^+}{x_3}-\frac{x_3}{x_3^+}\frac{x_5}{x_5^+}+\frac{x_5}{x_5^+}\right)\\
&\le \alpha_4 x_1^+\left(-2-\ln\frac{x_3^+}{x_3}-\ln\frac{x_1}{x_1^+}+\frac{x_3^+}{x_3}+\frac{x_1}{x_1^+}\right)+x_3^+(1+\beta_3)\left(2-\frac{x_3}{x_3^+}-\frac{x_3^+}{x_3}\right)\\
&+\alpha_1 x_3^+ x_5^+\left(1-\frac{x_3^+}{x_3}-\frac{x_3}{x_3^+}\frac{x_5}{x_5^+}+\frac{x_5}{x_5^+}\right)\\
&= \alpha_4 x_1^+(-E_1-\tilde{E}_3)+(1+\beta_3)x_3^+(E_3+\tilde{E}_3)\\
&+\alpha_1 x_3^+ x_5^+\left(1-\frac{x_3^+}{x_3}-\frac{x_3}{x_3^+}\frac{x_5}{x_5^+}+\frac{x_5}{x_5^+}\right),
\end{aligned}
$$

and lastly:

$$
\begin{aligned}
L_4'(t) &= \dot{x}_4\left(1-\frac{x_4^+}{x_4}\right)=\left(1-\frac{x_4^+}{x_4}\right)(x_1x_5+\alpha_2x_2x_5+\alpha_1x_3x_5-\alpha_3x_4-\beta_4x_4)\\
&= \left(1-\frac{x_4^+}{x_4}\right)\left[x_1^+x_5^+\left(\frac{x_1x_5}{x_1^+x_5^+}-1\right)+\alpha_2x_2^+x_5^+\left(\frac{x_2x_5}{x_2^+x_5^+}-1\right)\right.\\
&\left.+\alpha_1x_3^+x_5^+\left(\frac{x_3x_5}{x_3^+x_5^+}-1\right)-(\alpha_3+\beta_4)x_4^+\left(\frac{x_4}{x_4^+}-1\right)\right]\\
&= x_1^+x_5^+\left(\frac{x_1x_5}{x_1^+x_5^+}-1-\frac{x_4^+x_1x_5}{x_4x_1^+x_5^+}+\frac{x_4^+}{x_4}\right)\\
&+\alpha_2x_2^+x_5^+\left(\frac{x_2x_5}{x_2^+x_5^+}-1-\frac{x_4^+x_2x_5}{x_4x_2^+x_5^+}+\frac{x_4^+}{x_4}\right)\\
&+\alpha_1x_3^+x_5^+\left(\frac{x_3x_5}{x_3^+x_5^+}-1-\frac{x_4^+x_3x_5}{x_4x_3^+x_5^+}+\frac{x_4^+}{x_4}\right)\\
&+(\alpha_3+\beta_4)x_4^+\left(2-\frac{x_4}{x_4^+}-\frac{x_4^+}{x_4}\right)\\
&\le x_1^+x_5^+\left(\frac{x_1x_5}{x_1^+x_5^+}-2-\ln\frac{x_4^+}{x_4}-\ln\frac{x_1}{x_1^+}-\ln\frac{x_5}{x_5^+}+\frac{x_4^+}{x_4}\right)\\
&+\alpha_2x_2^+x_5^+\left(\frac{x_2x_5}{x_2^+x_5^+}-2-\ln\frac{x_4^+}{x_4}-\ln\frac{x_2}{x_2^+}-\ln\frac{x_5}{x_5^+}+\frac{x_4^+}{x_4}\right)\\
&+\alpha_1x_3^+x_5^+\left(\frac{x_3x_5}{x_3^+x_5^+}-2-\ln\frac{x_4^+}{x_4}-\ln\frac{x_3}{x_3^+}-\ln\frac{x_5}{x_5^+}+\frac{x_4^+}{x_4}\right)\\
&+(\alpha_3+\beta_4)x_4^+(E_4+\tilde{E}_4).
\end{aligned}
$$

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$$
\begin{aligned}
L_5'(t) &= \dot{x}_5\left(1-\frac{x_5^+}{x_5}\right) = \left(1-\frac{x_5^+}{x_5}\right)(x_4-\beta_5 x_5)\\
&= \left(1-\frac{x_5^+}{x_5}\right)(x_4-x_4^+-\beta_5 x_5+\beta_5 x_5^+)\\
&= \beta_5 x_5^+\left(2-\frac{x_5^+}{x_5}-\frac{x_5}{x_5^+}\right)+x_4^+\left(\frac{x_4}{x_4^+}-1-\frac{x_5^+ x_4}{x_5 x_4^+}+\frac{x_5^+}{x_5}\right)\\
&\le x_4^+\left(2-\frac{x_5^+}{x_5}-\frac{x_5}{x_5^+}\right)+x_4^+\left(\frac{x_4}{x_4^+}-2+\ln\frac{x_5}{x_5^+}-\ln\frac{x_4}{x_4^+}+\frac{x_5^+}{x_5}\right)\\
&= x_4^+\left(-\frac{x_5}{x_5^+}+\frac{x_4}{x_4^+}+\ln\frac{x_5}{x_5^+}-\ln\frac{x_4}{x_4^+}\right)\\
&= x_4^+(E_4-E_5)
\end{aligned}
$$

The following results are also obtained by considering the algebraic relations that the coordinates of the equilibrium point S^+ satisfies:

$$
\begin{aligned}
&L_1'(t)+L_2'(t)+L_3'(t)+L_4'(t)+L_5'(t)\le\\
&\le x_1^+x_5^+\Big(1-\frac{x_1^+}{x_1}+\frac{x_5}{x_5^+}-2+\ln\frac{x_1^+}{x_1}-\ln\frac{x_4^+}{x_4}-\ln\frac{x_5}{x_5^+}+\frac{x_4^+}{x_4}\Big)\\
&+\alpha_2 x_2^+x_5^+\left(1-\frac{x_2^+}{x_2}+\frac{x_5}{x_5^+}-2-\ln\frac{x_4^+}{x_4}+\ln\frac{x_2^+}{x_2}-\ln\frac{x_5}{x_5^+}+\frac{x_4^+}{x_4}\right)\\
&+\alpha_1 x_3^+x_5^+\left(1-\frac{x_3^+}{x_3}+\frac{x_5}{x_5^+}-2-\ln\frac{x_4^+}{x_4}+\ln\frac{x_3^+}{x_3}-\ln\frac{x_5}{x_5^+}+\frac{x_4^+}{x_4}\right)\\
&-\alpha_3 x_4^+(\tilde{E}_1+E_4)-x_3^+(\tilde{E}_1+E_3)+(\alpha_4+\beta_1)x_1^+(E_1+\tilde{E}_1)\\
&+\beta_2 x_2^+(\tilde{E}_2+E_2)-\alpha_4 x_1^+(E_1+\tilde{E}_3)+(1+\beta_3)x_3^+(E_3+\tilde{E}_3)\\
&+(\alpha_3+\beta_4)x_4^+(E_4+\tilde{E}_4)\\
&= x_1^+x_5^+(\tilde{E}_1-E_5-\tilde{E}_4)+\alpha_2 x_2^+x_5^+(\tilde{E}_2-E_5-\tilde{E}_4)\\
&+\alpha_1 x_3^+x_5^+(\tilde{E}_3-E_5-\tilde{E}_4)-\alpha_3 x_4^+(\tilde{E}_1+E_4)-x_3^+(\tilde{E}_1+E_3)\\
&+(\alpha_4+\beta_1)x_1^+(E_1+\tilde{E}_1)+\beta_2 x_2^+(\tilde{E}_2+E_2)-\alpha_4 x_1^+(E_1+\tilde{E}_3)\\
&+(1+\beta_3)x_3^+(E_3+\tilde{E}_3)+(\alpha_3+\beta_4)x_4^+(E_4+\tilde{E}_4)\\
&= \tilde{E}_1\left[x_1^+x_5^+-\alpha_3 x_4^+-x_3^++(\alpha_4+\beta_1)x_1^+\right]+\beta_1 x_1^+E_1\\
&+(\alpha_2 x_2^+x_5^++\beta_2 x_2^+)\tilde{E}_2+\beta_2 x_2^+E_2\\
&+\tilde{E}_3[\alpha_1 x_3^+x_5^+-\alpha_4 x_1^++(1+\beta_3)x_3^+]+\beta_3 x_3^+E_3\\
&+\tilde{E}_4[-x_1^+x_5^+-\alpha_2 x_2^+x_5^+-\alpha_1 x_3^+x_5^++(\alpha_3+\beta_4)x_4^+]+\beta_4 x_4^+E_4\\
&+E_5[-x_1^+x_5^+-\alpha_2 x_2^+x_5^+-\alpha_1 x_3^+x_5^+]
\end{aligned}
$$

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$$= \gamma_1 \tilde{E}_1 + \gamma_2 \tilde{E}_2 + \beta_1 x_1^+ E_1 + \beta_2 x_2^+ E_2 + \beta_3 x_3^+ E_3 + \beta_4 x_4^+ E_4 - (\alpha_3 + \beta_4) x_4^+ E_5.$$

□

6 Conclusions

In this paper we present a five-dimensional mathematical model that provides a new approach to studying the labour market, with a focus on unemployment levels, migration, fixed-term contractors, full-time employment, and the number of available vacancies. We analyse the positivity and boundedness of the solutions and discuss the existence of equilibrium points using the basic reproduction number. Local and global stability properties of the model are also demonstrated.

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