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THE ENERGY-CASIMIR MAPPING OF THE JERK VERSION OF AN ANHARMONIC OSCILLATOR

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Abstract

In this paper we study the stability and some special orbits of an one-parameter Hamilton-Poisson jerk system, namely the jerk version of an anharmonic oscillator. Moreover, we point out some properties of the energy-Casimir mapping associated to this system. [1]

1 Introduction

In a mechanical system, let $x = x(t)$ be the displacement of a moving object. Then, $\dot{x} = \dfrac{dx}{dt}$ and $\ddot{x} = \dfrac{d^2x}{dt^2}$ represents its velocity and acceleration, respectively. In addition, the jerk is the rate of change of acceleration, the third derivative $\dddot{x}$ of position with respect to time [7]. A differential equation of the form

$$\dddot{x} = j(x, \dot{x}, \ddot{x}),$$

where j is a smooth function usually, is called a jerk equation. This equation takes the form of a differential system, namely

$$\dot{x} = y, \dot{y} = z, \dot{z} = j(x, y, z),$$

called a jerk system.

The harmonic oscillator, described by the equation $\ddot{x} + \omega x^2 = 0$, models processes that exhibit sinusoidal oscillations with constant amplitude. Anharmonic oscillators are a type of oscillator that deviate from the simple harmonic motion. These oscillators are often described by equations that include nonlinear

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terms to account for the deviation from simple harmonic motion, and they can exhibit nonlinear behavior. Such an anharmonic oscillator is given by the equation $\ddot{x} + \delta x^n = 0$ [4].

In this paper we study the jerk version of the anharmonic oscillator

$$\ddot{x} - \frac{g}{3}x^3 = 0,$$

namely the jerk equation

$$\dddot{x} - gx^2\dot{x} = 0,$$

or equivalent, the jerk system

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = gx^2y \end{cases}, \tag{1}$$

where g is a real parameter.

The paper is organized as follows. In Section 2, we give the Hamilton-Poisson realization of the considered system. Then, following [8], we consider the energy-Casimir mapping associated to this realization. Moreover, we determine the image of this mapping. In Section 3, we point out some fibers of the energy-Casimir mapping in correspondence with some dynamical properties of the considered system, such as stable equilibrium points, periodic orbits, and so called split-homoclinic orbits.

2 Energy-Casimir mapping

In this section we point out a Hamilton-Poisson realization of the considered system and the image of the corresponding energy-Casimir mapping.

We consider the functions

$$H(x,y,z) = \frac{y^2}{2} - xz + \frac{gx^4}{4}\ , \quad C(x,y,z) = z - \frac{gx^3}{3}. \tag{2}$$

Using (1) we deduce $\dot{H} = 0$ and $\dot{C} = 0$, thus H and C are constants of motion. Moreover, $\Pi \cdot \nabla C = \mathbf{0}$ and system (1) writes $\dot{\mathbf{x}}^T = \Pi \cdot \nabla H$, where $\mathbf{x} = (x,y,z)^T$ and

$$\Pi = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -gx^2 \\ 0 & gx^2 & 0 \end{bmatrix}.$$

Consequently, system (1) has the Hamilton-Poisson realization $\left(\mathbb{R}^3, \Pi, H\right)$, where H is the Hamiltonian function and C is a Casimir of the Poisson structure given by Π.

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In this framework, the energy-Casimir mapping associated to system (1) is defined by $\mathcal{EC}:\mathbb{R}^3\to\mathbb{R}^2$,

$$\mathcal{EC}(x,y,z)=(H(x,y,z),C(x,y,z))=\left(\frac{y^2}{2}-xz+\frac{gx^4}{4},z-\frac{gx^3}{3}\right). \tag{3}$$

We notice that system (1) writes $\dot{\mathbf{x}}^T=\nabla H\times\nabla C$. Therefore the critical points of energy Casimir mapping (3) are the equilibrium points of system (1), namely $E_M=(M,0,0), M\in\mathbb{R}$.

The image of the energy-Casimir mapping $\mathcal{EC}$ is the set

$$\mathrm{Im}(\mathcal{EC})=\left\{(h,c)\in\mathbb{R}^2|(\exists)(x,y,z)\in\mathbb{R}^3:\mathcal{EC}(x,y,z)=(h,c)\right\}.$$

As we have seen in other papers ([8]; also see [5] and references therein), the image $\mathcal{EC}(E_M)$ of the critical points through the energy-Casimir mapping leads to a partition of $\mathrm{Im}(\mathcal{EC})$. In our case, $\mathcal{EC}(E_M)$ is the curve

$$\Gamma=\left\{(h,c)\in\mathbb{R}^2|h=\frac{gM^4}{4},c=-\frac{gM^3}{3},M\in\mathbb{R}\right\}. \tag{4}$$

For $g\neq 0$, the curve Γ takes the form

$$\Gamma=\left\{(h,c)\in\mathbb{R}^2|c^4-\frac{64g}{81}h^3=0\right\}. \tag{5}$$

We define the sets

$$\Sigma_1=\left\{(h,c)\in\mathbb{R}^2|c^4-\frac{64g}{81}h^3<0\right\}, \tag{6}$$

$$\Sigma_2=\left\{(h,c)\in\mathbb{R}^2|c^4-\frac{64g}{81}h^3>0\right\}. \tag{7}$$

Proposition 2.1. *Let $g>0$ and let $\mathcal{EC}$ (3) be the energy-Casimir mapping of system* (1)*. Then Im$(\mathcal{EC})=\Gamma\cup\Sigma_1\cup\Sigma_2=\mathbb{R}^2$ (Figure 2).*

Proof. A pair (h,c) belongs to the image of the energy-Casimir mapping if and only if the system

$$\begin{cases} h=\dfrac{y^2}{2}-xz+\dfrac{gx^4}{4} \\ c=z-\dfrac{gx^3}{3} \end{cases} \tag{8}$$

has at least a solution. Let us denote $h_e=\frac{gM^4}{4}$ and $c_e=-\frac{gM^3}{3}$, for any $M\in\mathbb{R}$.

Let $(h,c)\in\Gamma\cup\Sigma_1$ such that $h\geq h_e$. Using (8) we get

$$y^2=\frac{gx^4}{6}+2cx+2h,$$

or equivalent

$$y^2=\frac{g}{6}(x-M)^2[(x+M)^2+2M^2]+2(h-h_e).$$

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Because $h \geq h_e$, system (8) has solution. Therefore, $\Gamma \cup \Sigma_1 \subset \mathrm{Im}(\mathcal{EC})$.

Let $(h, c) \in \Sigma_2$ and $y = 0$. Using (8) we obtain

$$\frac{gx^4}{6} + 2cx + 2h = 0.$$

We will deduce that system (8) has solution by showing the above equation (2) has solution for $(h, c) \in \Sigma_2$. Consider the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \frac{gx^4}{6} + 2cx + 2h.$$

By studying the behavior of the function f, we have $f_{min} = 2h - \frac{3c}{2}\sqrt[3]{\frac{3c}{g}} < 0$ for any $(h, c) \in \Sigma_2$, hence the equation (2) has solution if $(h, c) \in \Sigma_2$. Therefore, system (8) has solution for $(h, c) \in \Sigma_2$. Thus $\Sigma_2 \subset \mathrm{Im}(\mathcal{EC})$ and consequently $\mathrm{Im}(\mathcal{EC}) = \mathbb{R}^2$, as required. □

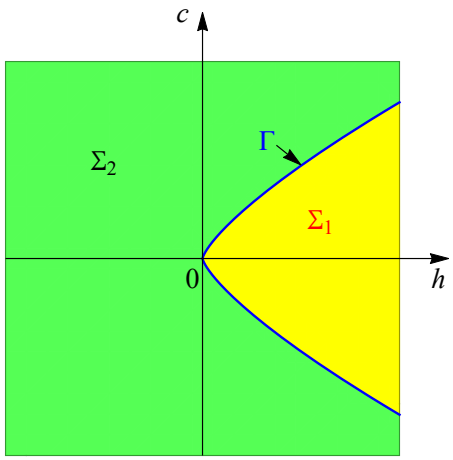


Figure 2: The image of the energy-Casimir mapping ($g > 0$).

Proposition 2.2. *Let $g < 0$ and let $\mathcal{EC}$ (3) be the energy-Casimir mapping of system (1). Then $Im(\mathcal{EC}) = \Gamma \cup \Sigma_2$ (Figure 3).*

Proof. It is clear that system (8) has solution for $h = h_e = \frac{gM^4}{4}$ and $c = c_e = -\frac{gM^3}{3}$, for every $M \in \mathbb{R}$.

First, we prove that there are no pairs $(h, c) \in \mathrm{Im}(\mathcal{EC})$ for $(h, c) \in \Sigma_1$ by showing that the system (8) has no solution. Suppose that system (8) has solution for $h < h_e$ and $c = c_e$. Using (8) we obtain

$$-g(x - M)^2[(x + M)^2 + 2M^2] + 6y^2 = 12\left(h - \frac{gM^4}{4}\right)$$

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or equivalent

$$-g(x-M)^2[(x+M)^2+2M^2]+6y^2=12\,(h-h_e)\,. \tag{9}$$

But $h<h_e$, which leads to a contradiction. Thus, for $h<h_e$ and $c=c_e$ system (8) has no solution, that is $\Sigma_1\cap \mathrm{Im}(\mathcal{EC})=\emptyset$.

Now, let $(h,c)\in\Sigma_2$. Consider $y=0$ and the function f defined in (2). Using the similar method as in Proposition 2.1, we get that system (8) has solution for $(h,c)\in\Sigma_2$. Therefore $\Sigma_2\subset \mathrm{Im}(\mathcal{EC})$, and the conclusion follows. □

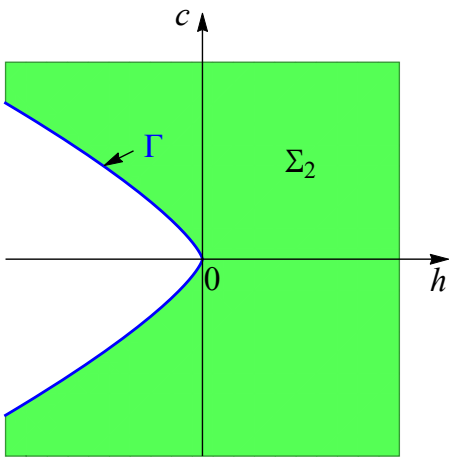


Figure 3: The image of the energy-Casimir mapping ($g<0$).

Proposition 2.3. *Let $g=0$ and let $\mathcal{EC}$ (3) be the energy-Casimir mapping of system (1). Then $Im(\mathcal{EC})=\mathbb{R}^2\backslash\Sigma_0$, where $\Sigma_0=\{(h,c)\in\mathbb{R}^2|h<0,c=0\}$ (Figure 4).*

Proof. Let $(h,c)\in\Sigma_0$. Suppose that system (8) has solution for $h<0$ and $c=0$. System (8) becomes

$$\begin{cases} h=\dfrac{y^2}{2}-xz \\ z=0 \end{cases} \tag{10}$$

From (10) it follows that $h=\dfrac{y^2}{2}>0$ which leads to a contradiction. Thus, for $h<0$ and $c=0$ system (8) has no solution. Therefore $\Sigma_0\cap\mathrm{Im}(\mathcal{EC})=\emptyset$.

If $(h,c)\in\mathbb{R}^2\backslash\Sigma_0$, system (8) becomes

$$\begin{cases} h=\dfrac{y^2}{2}-xz \\ c=z \end{cases}. \tag{11}$$

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Using (11) we get $cx+h=\dfrac{y^2}{2}>0$. There exist $x\in\mathbb{R}$ such that $cx+h>0$ for all $(h,c)\in\mathbb{R}^2\backslash\Sigma_0$, thus system (8) has solutions. Therefore, $\mathbb{R}^2\backslash\Sigma_0\subset \mathrm{Im}(\mathcal{EC})$ and the conclusion follows. □

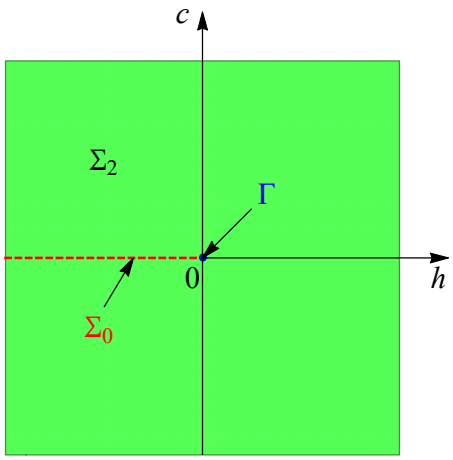


Figure 4: The image of the energy-Casimir mapping ($g=0$).

3 Connections between the energy-Casimir mapping and some dynamical properties of the system

In this section we study the stability of the equilibrium points, the existence of periodic orbits and other types of orbits of system (1) in connection with the image of the energy-Casimir mapping (3).

Proposition 3.1. *Let $g<0$. Then the equilibrium point $E_M=(M,0,0)$, $M\in\mathbb{R}$ is nonlinearly stable.*

Proof. The jacobian matrix of system (1) at E_M is

$$J(M,0,0)=\begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & gM^2 & 0\end{bmatrix}. \tag{12}$$

The corresponding eigenvalues are $\lambda_1=0,\lambda_{2,3}=\pm M\sqrt{g}$.

Let $M\neq 0$. Then the above Jacobian function (12) has a pair of imaginary eigenvalues. In this case we use the Arnold stability test [1]. We consider the

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function

$$F(x,y,z) = H(x,y,z) + MC(x,y,z). \tag{13}$$

The following hold:

1. $dF(M,0,0) = 0$.

2. $W = \ker dC(M,0,0) = \text{span}_{\mathbb{R}}\{(1,0,gM^2),(0,1,0)\}$.

3. $d^2F(M,0,0)|_{W\times W} = dy^2 - \dfrac{1}{gM^2}dz^2$ is positive definite ($g<0$ and $M\neq 0$).

In conclusion, E_M is a nonlinear stable (Lyapunov stable) equilibrium point for $M \neq 0$ in the case $g < 0$.

For $M = 0$, the equilibrium point E_M becomes $(0,0,0)$. We show that the function $L \in C^\infty(\mathbb{R}^3,\mathbb{R})$,

$$L(x,y,z) = \left(\frac{y^2}{2} - xz + \frac{gx^4}{4}\right)^2 + \left(z - \frac{gx^3}{3}\right)^2 \tag{14}$$

is a Lyapunov function for system (1) and the equilibrium $(0,0,0)$.

It is easy to check that $\dot{L} = \nabla L \cdot \dot{\mathbf{x}} = 0$, where $\dot{\mathbf{x}} = (\dot{x},\dot{y},\dot{z})$ is given by (1). In addition, the condition $L(x,y,z) = 0$ implies $\dfrac{y^2}{2} - xz + \dfrac{gx^4}{4} = 0, z - \dfrac{gx^3}{3} = 0$, whence $-gx^4+6y^2 = 0$. Because $g < 0$, it follows that $x = y = 0$ and consequently $z = 0$. Thus $L(x,y,z) > 0$, $\forall (x,y,z) \neq (0,0,0)$ and $L(0,0,0) = 0$. In conclusion, L is a Lyapunov function and the equilibrium point $(0,0,0)$ is nonlinearly stable, which finishes the prove. □

Remark 3.2. *For $g < 0$, the images of the stable equilibrium points $(M,0,0)$ through the energy-Casimir mapping (3) give the border Γ (5) of the set Im($\mathcal{EC}$) (see Figure 3).*

The inverse image of $(h,c) \in \text{Im}(\mathcal{EC})$ under the energy-Casimir mapping $\mathcal{EC}$ is the set

$$\mathcal{F}_{(h,c)} = \left\{(x,y,z) \in \mathbb{R}^3 \mid \mathcal{EC}(x,y,z) = (h,c)\right\}$$

called the fiber of $\mathcal{EC}$ corresponding to (h,c).

Proposition 3.3. *Let $g < 0$ and $(h,c) \in \Gamma$ (4). Then the corresponding fiber $\mathcal{F}_{(h,c)}$ is a nonlinear stable equilibrium point, namely*

$$\mathcal{F}_{(h,c)} = \left\{(M,0,0)\,|M = -\sqrt[3]{\frac{3c}{g}}\right\}.$$

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Proof. Let $(h,c) \in \Gamma$ (4). Then there is $M \in \mathbb{R}$ such that $h = \dfrac{gM^4}{4}$, $c = -\dfrac{gM^3}{3}$, namely $M = -\sqrt[3]{\dfrac{3c}{g}}$. The condition $\mathcal{EC}(x,y,z) = (h,c)$ is equivalent to

$$\begin{cases} \dfrac{y^2}{2} - xz + \dfrac{gx^4}{4} = \dfrac{gM^4}{4} \\ z - \dfrac{gx^3}{3} = -\dfrac{gM^3}{3} \end{cases}.$$

We get

$$-g(x-M)^2[(x+M)^2 + 2M^2] + 6y^2 = 0,$$

and taking into account that $g < 0$, it follows $x = M$, $y = z = 0$, as required. □

We expect that there are periodic orbits around each stable equilibrium point $(M,0,0)$. Indeed, we have.

Proposition 3.4. *Let $g < 0$ and $M \in \mathbb{R}^*$. Then for each sufficiently small $\varepsilon \in \mathbb{R}_+^*$, any integral surface*

$$\Sigma_\varepsilon^M : \frac{g}{12}(3x^4 - 4Mx^3 + M^4) + \frac{1}{2}y^2 - (x-M)z = \varepsilon^2$$

contains at least one periodic orbit γ_ε^M of system (1) whose period is close to $\dfrac{2\pi}{\omega}$, where $\omega = M\sqrt{-g}$.

Proof. The characteristic polynomial of the Jacobian matrix (12) of system (1) at the stable equilibrium point $E_M = (M,0,0)$ has the eigenvalues $\lambda_1 = 0$ and $\lambda_{2,3} = \pm iM\sqrt{-g}$. Hence we can apply a version of the Moser theorem regarding the existence of periodic orbits in the case of a zero eigenvalue [2].

The eigenspace corresponding to the eigenvalue zero, which is $\operatorname{span}_{\mathbb{R}}\{(1,0,0)\}$, has dimension 1.

Let us consider the constant of motion of system (1)

$$I(x,y,z) = \frac{y^2}{2} - xz + \frac{gx^4}{4} + M\left(z - \frac{gx^3}{3}\right). \tag{15}$$

It follows that:

1. $dI(M,0,0) = 0$.

2. $d^2I(M,0,0)|_{W\times W} = dy^2 - \dfrac{1}{gM^2}dz^2 > 0$ is positive definite $(g < 0, M \neq 0)$, where $W = \ker dC(M,0,0) = \operatorname{span}_{\mathbb{R}}\{(1,0,gM^2),(0,1,0)\}$.

Using the above-mentioned theorem, the conclusion follows. □

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Remark 3.5. *Let $g < 0$ and $(h, c) \in \Sigma_2$, i.e. (h, c) belongs to the interior of the image of $\mathcal{EC}$. Then the above proposition tells us that as long as the intersection of the level sets $H(x, y, z) = h, C(x, y, z) = c$ is a closed curve, the corresponding fiber $\mathcal{F}_{(h,c)}$ is a periodic orbit around the stable equilibrium point E_M (Figure 5). Moreover, when $\varepsilon \to 0$ these curves shrink to E_M. Such periodic orbits are presented in Figure 6, considering c fixed and h variable.*

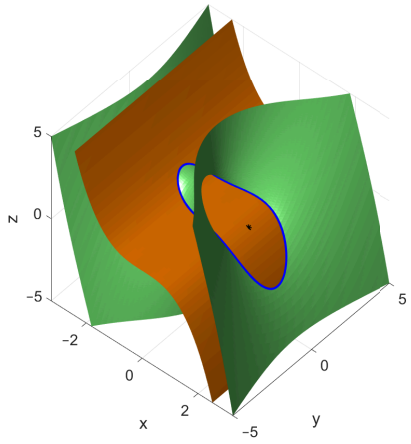


Figure 5: The fiber $\mathcal{F}_{(h,c)}$, $(h, c) \in \Sigma_2$: a periodic orbit.

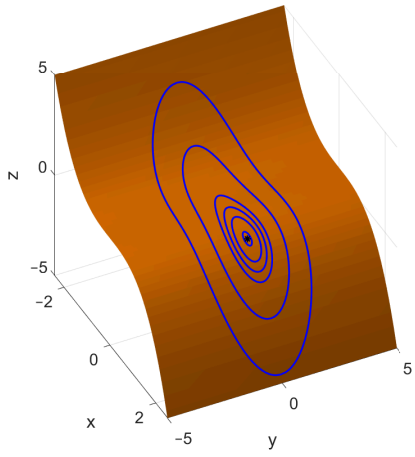


Figure 6: The family of periodic orbits around a stable equilibrium point (on the level set $C(x, y, z) = c$).

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Proposition 3.6. *Let $g \geq 0$. Then the equilibrium point $E_M = (M, 0, 0)$, $M \in \mathbb{R}$ is unstable.*

Proof. If $g = 0$, system (1) becomes the equation $\dddot{x} = 0$, which has the unbounded solution $x(t) = \dfrac{C_1}{2}t^2 + C_2 t + C_3$, where $C_1, C_2, C_3 \in \mathbb{R}$. It results that E_M is an unstable equilibrium point.

Now, let $g > 0$. As we have seen, the eigenvalues of the jacobian matrix at E_M (12) are $\lambda_1 = 0, \lambda_{2,3} = \pm M\sqrt{g}$. Thus, if $M \neq 0$, then $\lambda_{2,3}$ is a pair of opposite sign real eigenvalues, thus E_M is unstable.

For $M = 0$, the eigenvalues the Jacobian function (12) are $\lambda_1 = \lambda_2 = \lambda_3 = 0$. In this case we determine a solution of system (1) that starts out near $(0, 0, 0)$, but it is unbounded, i.e. we prove that $(0, 0, 0)$ is unstable.

The dynamics of system (1) takes place at the intersection of the level sets $H(x, y, z) = constant, C(x, y, z) = constant$. We consider

$$H(x, y, z) = H(0, 0, 0), C(x, y, z) = C(0, 0, 0),$$

which is equivalent with

$$\begin{cases} \dfrac{y^2}{2} - xz + \dfrac{gx^4}{4} = 0 \\ z - \dfrac{gx^3}{3} = 0 \end{cases} . \tag{16}$$

We chose an initial condition (x_0, y_0, z_0) in a neighborhood of $(0, 0, 0)$ that satisfies (16), namely $x_0 = \varepsilon$, $y_0 = a\varepsilon^2$, $z_0 = 2a^2\varepsilon^3$, where $\varepsilon > 0$ is close to zero and $a = \sqrt{\dfrac{g}{6}}$. Then there is a solution $(x(t), y(t), z(t))$ with $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$ which also satisfies (16). We get $y(t) = ax^2(t), z(t) = 2a^2x^3(t)$ and system (1) reduces in this case to the equation

$$\dot{x} = ax^2, a > 0.$$

We obtain the solution

$$x(t) = \frac{\varepsilon}{1 - a\varepsilon t} , \quad y(t) = \frac{a\varepsilon^2}{(1 - a\varepsilon t)^2} , \quad z(t) = \frac{2a^2\varepsilon^3}{(1 - a\varepsilon t)^3}, \quad t \in \left[0, \frac{1}{a\varepsilon}\right),$$

where $\varepsilon > 0$ is close to zero and $a = \sqrt{\dfrac{g}{6}}$.

It easy to see that this solution starts near equilibrium point $(0, 0, 0)$, but it does not stay near $(0, 0, 0)$. Therefore, the equilibrium point $(0, 0, 0)$ is unstable, which finishes the prove. □

For $g > 0$, the image of the energy-Casimir mapping is $\mathbb{R}^2$. The images of the unstable equilibrium points $(M, 0, 0)$ through $\mathcal{EC}$ belong to the curve Γ (5), and the corresponding fibers can contain homoclinic or heteroclinic orbits (see,

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e.g., [3]), or split-homoclinic and split-heteroclinic orbits [6]. In our case we have obtained split-homoclinic orbits.

Recall that a homoclinic orbit $\mathcal{H} : \mathbb{R} \to \mathbb{R}^3$ is a solution $(x(t), y(t), z(t))$ of the considered system which joins an unstable equilibrium point E to itself, that is $\mathcal{H}(t) := (x(t), y(t), z(t))$ and $\mathcal{H}(t) \to E$ as $t \to \pm\infty$. In addition, if $(x(t), y(t), z(t))$, $t \in (-\infty, s) \cup (s, \infty)$ is a solution of the considered system, which is indefinite in s, such that $(x(t), y(t), z(t)) \to E$ as $t \to \pm\infty$, then we say that $\mathcal{SH} : (-\infty, s) \cup (s, \infty) \to \mathbb{R}^3$, $\mathcal{SH}(t) := (x(t), y(t), z(t))$ is a split-homoclinic orbit.

Proposition 3.7. *Let $g > 0$, $(h, c) \in \Gamma, c \neq 0$, and $M = \sqrt[3]{\dfrac{-3c}{g}}$. Denote*

$$
\begin{aligned}
x_1(t) &= \frac{M\left(e^{2p(t)} + 8e^{p(t)} - 2\right)}{e^{2p(t)} - 4e^{p(t)} - 2},\\
y_1(t) &= -\frac{12M^2 e^{p(t)}\sqrt{g}\left(e^{2p(t)} + 2\right)}{\left(e^{2p(t)} - 4e^{p(t)} - 2\right)^2},\\
z_1(t) &= \frac{gM^3}{3}\left(\frac{\left(e^{2p(t)} + 8e^{p(t)} - 2\right)^3}{\left(e^{2p(t)} - 4e^{p(t)} - 2\right)^3} - 1\right),
\end{aligned}
$$

$$
\begin{aligned}
x_2(t) &= \frac{M\left(2e^{2p(t)} - 8e^{p(t)} - 1\right)}{2e^{2p(t)} + 4e^{p(t)} - 1},\\
y_2(t) &= \frac{12M^2 e^{p(t)}\sqrt{g}\left(2e^{2p(t)} + 1\right)}{\left(2e^{2p(t)} + 4e^{p(t)} - 1\right)^2},\\
z_2(t) &= \frac{gM^3}{3}\left(\frac{\left(2e^{2p(t)} - 8e^{p(t)} - 1\right)^3}{\left(2e^{2p(t)} + 4e^{p(t)} - 1\right)^3} - 1\right),
\end{aligned}
$$

with $p(t) = M(t\sqrt{g} + K)$ and $K \in \mathbb{R}$.

Then the fiber $\mathcal{F}_{(h,c)}$ contains two split-homoclinic orbits

$$
\begin{aligned}
\mathcal{SH}_1 &: \mathbb{R} \setminus \left\{\frac{\ln\left(2 + \sqrt{6}\right) - KM}{M\sqrt{g}}\right\} \to \mathbb{R}^3, \mathcal{SH}_1 = (x_1, y_1, z_1),\\
\mathcal{SH}_2 &: \mathbb{R} \setminus \left\{\frac{-KM - \ln(2) + \ln\left(\sqrt{6} - 2\right)}{M\sqrt{g}}\right\} \to \mathbb{R}^3, \mathcal{SH}_2 = (x_2, y_2, z_2),
\end{aligned}
$$

which tend to the equilibrium point $E_M = (M, 0, 0)$ as $t \to \pm\infty$

Proof. The intersection of the level sets $H(x, y, z) = h, C(x, y, z) = c$ for $(h, c) \in \Gamma$ is given by

$$
\begin{cases} H(x, y, z) = H(M, 0, 0) \\ C(x, y, z) = C(M, 0, 0) \end{cases},
$$

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or equivalent

$$\begin{cases} \dfrac{y^2}{2} - xz + \dfrac{gx^4}{4} = \dfrac{gM^4}{4} \\ z - \dfrac{gx^3}{3} = -\dfrac{gM^3}{3} \end{cases}.$$

We get

$$z = \frac{gx^3}{3} - \frac{gM^3}{3}, y^2 = \frac{1}{6}g(x-M)^2[(x+M)^2 + 2M^2]. \tag{17}$$

System (1) reduces in this case to the equation $\dot{x} = y$. First, we consider

$$\dot{x} = -\sqrt{\frac{g}{6}}(x-M)\sqrt{(x+M)^2+2M^2},$$

which is equivalent with

$$\int \frac{dx}{(x-M)\sqrt{(x+M)^2+2M^2}} = -\sqrt{\frac{g}{6}}t + K,\ K \in \mathbb{R}.$$

By integration, we obtain the solution x_1, and using (17), the functions y_1 and z_1, with $t \in \mathbb{R}\setminus\left\{\dfrac{\ln\left(2+\sqrt{6}\right) - KM}{M\sqrt{g}}\right\}$.

Similarly, we reduce system (1) to

$$\dot{x} = \sqrt{\frac{g}{6}}(x-M)\sqrt{(x+M)^2+2M^2},$$

which is equivalent with

$$\int \frac{dx}{(x-M)\sqrt{(x+M)^2+2M^2}} = \sqrt{\frac{g}{6}}t + K,\ K \in \mathbb{R},$$

and we get the solution (x_2, y_2, z_2), $t \in \mathbb{R} \setminus \left\{\dfrac{-KM - \ln(2) + \ln\left(\sqrt{6}-2\right)}{M\sqrt{g}}\right\}$.

We can also see that

$$\lim_{t\to\pm\infty}(x_1(t), y_1(t), z_1(t)) = (M, 0, 0),$$
$$\lim_{t\to\pm\infty}(x_2(t), y_2(t), z_2(t)) = (M, 0, 0),$$

as required. □

Remark 3.8. *In Figure 7, some fibers of the energy-Casimir mapping* (3) *are drawn on the level set $C(x, y, z) = c$, c fixed. More precisely, the fiber corresponding to $(h, c) = \mathcal{EC}(M, 0, 0) \in \Gamma$ contains a pair of split-homoclinic orbits (white curves). Keeping c fixed and changing h, we obtain unbounded orbits (yellow curves for $(h, c) \in \Sigma_1$ and green curves for $(h, c) \in \Sigma_2$).*

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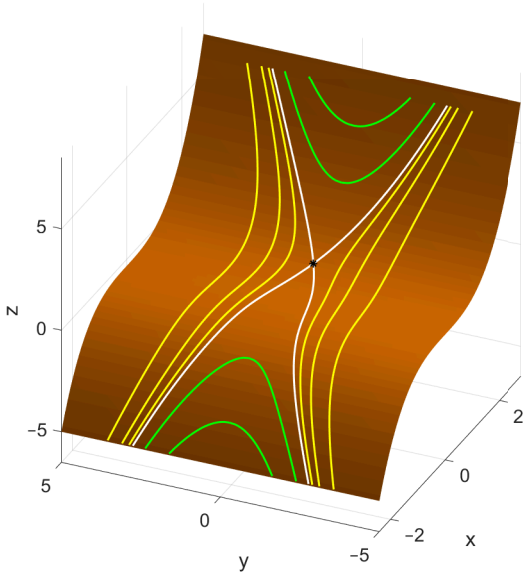


Figure 7: Split-homoclinic and other orbits near the unstable equilibrium point $E_M = (M, 0, 0)$, $M = \sqrt[3]{\dfrac{-3c}{g}}$, on the level set $C(x, y, z) = c$.

Proposition 3.9. *Let $g > 0$ and $x_0 \in \mathbb{R}^*$. Denote*

$$x_3(t) = \frac{x_0}{1 - ax_0t}\ , \quad y_3(t) = \frac{ax_0^2}{(1 - ax_0t)^2}\ , \quad z_3(t) = \frac{2a^2x_0^3}{(1 - ax_0t)^3},$$

$$x_4(t) = \frac{x_0}{1 + ax_0t}\ , \quad y_4(t) = \frac{-ax_0^2}{(1 + ax_0t)^2}\ , \quad z_4(t) = \frac{2a^2x_0^3}{(1 + ax_0t)^3},$$

where $a = \sqrt{\dfrac{g}{6}}$. Then the fiber $\mathcal{F}_{(0,0)}$ contains two split-homoclinic orbits

$$\mathcal{SH}_3 : \left(-\infty, \frac{1}{ax_0}\right) \cup \left(\frac{1}{ax_0}, \infty\right) \to \mathbb{R}^3, \quad \mathcal{SH}_3 = (x_3, y_3, z_3),$$

$$\mathcal{SH}_4 : \left(-\infty, -\frac{1}{ax_0}\right) \cup \left(-\frac{1}{ax_0}, +\infty\right) \to \mathbb{R}^3, \quad \mathcal{SH}_4 = (x_4, y_4, z_4),$$

which tend to the equilibrium point $E_0 = (0, 0, 0)$ as $t \to \pm\infty$ (Figure 8).

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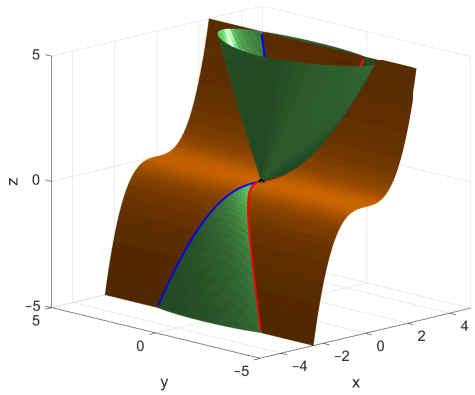


Figure 8: A pair of split-homoclinic orbits.

Proof. We consider the level sets

$$\begin{cases} H(x,y,z) = H(0,0,0) \\ C(x,y,z) = C(0,0,0) \end{cases},$$

that is

$$\begin{cases} \dfrac{y^2}{2} - xz + \dfrac{gx^4}{4} = 0 \\ z - \dfrac{gx^3}{3} = 0 \end{cases}.$$

Then

$$z = \frac{gx^3}{3}, y^2 = \frac{1}{6}x^4. \tag{18}$$

We chose an initial condition (x_0, y_0, z_0) that satisfies (18), that is $y_0 = ax_0^2$, $z_0 = 2a^2x_0^3$. System (1) reduces in this case to the equation

$$\dot{x} = ax^2, a > 0.$$

We obtain the solution x_3, and then y_3 and z_3, $t \in \mathbb{R}\backslash\left\{\dfrac{1}{ax_0}\right\}$.

Similarly, choosing $y_0 = -ax_0^2$, $z_0 = 2a^2x_0^3$, we reduce system (1) to

$$\dot{x} = -ax^2, a > 0,$$

and we get the solution x_4, and then y_4 and z_4, $t \in \mathbb{R}\backslash\left\{-\dfrac{1}{ax_0}\right\}$.

We can also see that

$$\lim_{t\to\pm\infty}(x_3(t), y_3(t), z_3(t)) = (0,0,0),$$
$$\lim_{t\to\pm\infty}(x_4(t), y_4(t), z_4(t)) = (0,0,0),$$

as required. □

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References

[1] V. Arnold, Conditions for nonlinear stability of stationary plane curvilinear flows on an ideal fluid, *Doklady Akademii Nauk SSSR*, **162**(5), (1965), 773–777.

[2] P. Birtea, M. Puta, R. M. Tudoran, Periodic orbits in the case of zero eigenvalue, *C.R. Acad. Sci. Paris*, Ser. I, **344**, (2007), 779–784.

[3] T. Bînzar, C. Lăzureanu, On some dynamical and geometrical properties of the Maxwell-Bloch equations with a quadratic control, *Journal of Geometry and Physics*, **70**, (2013), 1–8.

[4] J. Giné, D.I. Sinelshchikov, On the geometric and analytical properties of the anharmonic oscillator. *Communications in Nonlinear Science and Numerical Simulation*, **131**, (2024), 107875.

[5] C. Lăzureanu, On a Hamilton–Poisson approach of the Maxwell–Bloch equations with a control, *Mathematical Physics, Analysis and Geometry*, (2017), 20: 20.

[6] C. Lăzureanu, C. Hedrea, C. Petrişor, On a deformed version of the two-disk dynamo system, *Applications of Mathematics*, **66**(3), (2021), 345–372.

[7] Schot, S.H. Jerk: The time rate of change of acceleration. *Am. J. Phys.* **1978**, *46*, 1090–1094.

[8] R.M. Tudoran, A. Aron, Ş. Nicoară, On a Hamiltonian Version of the Rikitake System, *SIAM Journal on Applied Dynamical Systems*, **8**(1), (2009), 454–479.

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