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CURRENT DISTRIBUTION IN A MASSIVE COATED HOLLOW CONDUCTOR IN THE PRESENCE OF A FILAMENT

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Abstract

In this paper we present separate rigorous analyses of the skin and proximity effects in an inhomogeneous conductor in the presence of a filament. The conductor consists of a massive hollow circular conductor coated with a thin layer of different conductivity. The skin and proximity solutions are assumed in the form of two infinite sums of the proper harmonics. The unknown coefficients in the skin-effect case are found by applying boundary conditions, while a system of two integral equations is used to determine the unknown coefficients in the proximity-effect case, with no boundary conditions involved. By using the found current density we derive formulas for the conductor power loss.[1]

1 Introduction

There are very few cases where a solution for the current distribution of time-varying currents can be obtained in a closed form. These cases include some conductor configurations in the presence of a filament - massive circular conductor [1-3], thin tubular conductor [4,5], hollow massive conductor [2,6] thin two-layer tubular conductor [7] and inhomogeneous massive solid conductor [8]. The method of integral equations proved to be very powerful in these cases, requiring no boundary conditions when the conductor is homogeneous.

In this paper we investigate the skin and proximity effects in the case of a massive solid hollow circular conductor covered by a thin layer, in the presence of a filament. Although the conductor is inhomogeneous, no boundary conditions are required when analyzing the proximity part of the problem.

[1]Keywords and phrases: skin effect, proximity effect, inhomogeneous conductor, filament, current distribution, integral equation.

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2 Formulation of the problem and general form of the solution

Geometry of the problem is shown in Fig. 1. An inhomogeneous conductor consisting of a massive hollow conductor of radii a and b and conductivity σ_1, coated with a thin layer of thickness $d(d << a)$, and conductivity σ_2, and a filament carry equal and opposite sinusoidal currents of rms I and frequency f. Distance between the conductor axis and the filament is D. The object is to find current distribution in the conductor.

The proper radial harmonics in cylindrical coordinates are modified Bessel functions $I_{\mathrm{n}}(kr)$ and $K_{\mathrm{n}}(kr)$ $\left(k^2 = j\omega\mu_0\sigma\right)$, and the proper angular harmonics are trigonometrical functions $\cos n\theta$ and $\sin n\theta$. Due to symmetry, the sine function must be excluded, since the current density should be an even function of θ. Hence, the general form of solution for current density in region 1 is

$$J_1(r,\theta) = \sum_{n=0}^{\infty} \left[A_n I_n\left(k_1 r\right) + B_n K_n\left(k_1 r\right)\right] \cos n\theta \tag{1}$$

In region 2, due to its small thickness, the radial dependence may be neglected, so that general form of solution in this region is

$$J_2(\theta) = \sum_{n=0}^{\infty} C_n \cos n\theta \tag{2}$$

In (1) - (2), A_n, B_n and C_n are unknown coefficients that should be determined.

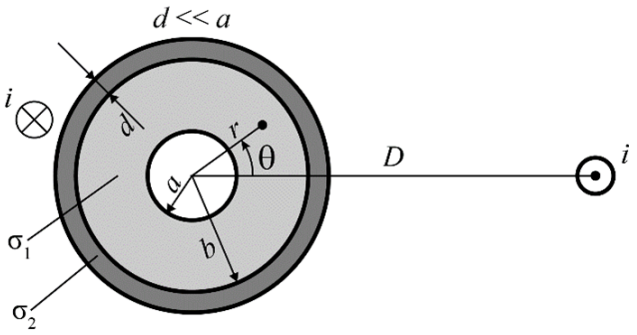


Figure 1: Inhomogeneous hollow conductor and fiament with equal and opposite currents

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It is convenient to treat separately the skin and proximity effects in this problem. The skin-effect solutions (the filament is absent) is represented by the first terms ($n = 0$) in (1) − (2), while the remaining infinite sums ($n \geq 1$) account for proximity effect.

2.1 Skin-effect solution

As mentioned above, this solution is given by

$$J_1^S(r) = A_0 I_0(k_1 r) + B_0 K_0(k_1 r) \tag{3}$$

$$J_2^S(r) = C_0 = \text{ const.} \tag{4}$$

The unknown coefficients A_0, B_0 and C_0 should be found from the boundary conditions and the known current in the conductor.

Equality of the tangential components of the electrical fields at the interface $r = b$ requires that

$$\frac{1}{\sigma_1}\left(A_0 I_0(k_1 b) + B_0 K_0(k_1 b)\right) = \frac{C_0}{\sigma_2} \tag{5}$$

The (tangential) magnetic field at $r = a$ should be zero, hence

$$A_0 I_1(k_1 a) - B_0 K_1(k_1 a) = 0 \tag{6}$$

The total current through the conductor is

$$I = \int_{S_1} J_1(r,\theta) r dr d\theta + bd \int_0^{2\pi} J_2(\theta) d\theta$$

where only the first terms ($n = 0$) from (1) - (2) should be taken, since $\int_0^{2\pi} \cos n\theta d\theta = 0, n \geq 1$. Therefore,

$$I = 2\pi \int_a^b \left[A_0 I_0(k_1 r) + B_0 K_0(k_1 r)\, r dr\right] + 2\pi b d C_0.$$

After integration we get

$$\frac{I k_1}{2\pi} = M_0 A_0 + N_0 B_0 + k_1 b d C_0 \tag{7}$$

where

$$M_0 = b I_1(k_1 b) - a I_1(k_1 a)$$
$$N_0 = a K_1(k_1 a) - b K_1(k_1 b)$$

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From (5) - (7) we can find the unknown coefficients

$$A_0 = \frac{Ik_1^2}{2\pi b} \frac{K_1(k_1 a)}{k_1 P_0 + k_2^2 d Q_0} \tag{8}$$

$$B_0 = \frac{Ik_1^2}{2\pi b} \frac{I_1(k_1 a)}{k_1 P_0 + k_2^2 d Q_0} \tag{9}$$

$$C_0 = \frac{Ik_2^2}{2\pi b} \frac{Q_0}{k_1 P_0 + k_2^2 d Q_0} \tag{10}$$

where

$$P_0 = I_1(k_1 b) K_1(k_1 a) - I_1(k_1 a) K_1(k_1 b)$$
$$Q_0 = I_0(k_1 b) K_1(k_1 a) + I_1(k_1 a) K_0(k_1 b)$$

Thus, the skin - effect solution, given by (3) - (4) is completed by (8) - (10).

2.2 Proximity-effect solution

To find the unknown coefficients A_n, B_n and $C_n (n \geq 1)$ in the proximity effect solutions

$$J_1^p(r,\theta) = \sum_{n=1}^{\infty} \left[A_n I_n(k_1 r) + B_n K_n(k_1 r)\right] \cos n\theta \tag{11}$$

$$J_2^p(\theta) = \sum_{n=1}^{\infty} C_n \cos n\theta \tag{12}$$

we use the method of integral equations. For the case of two conductors in the presence of a filament these equations have the following form in cylindrical coordinates [9].

$$J_1(r,\theta) = \frac{k_1^2}{4\pi} \left[\sum_{i=1}^{2} \int_{S_i} J_i\left(r',\theta'\right) \ln f\left(r,r',\theta,\theta'\right) dSi - \right. \tag{13}$$
$$\left. - I \ln f(r,r_0,\theta,\theta_0)\right] + k_1, \quad (r,\theta) \in S_1$$

$$J_2(r,\theta) = \frac{k_2^2}{4\pi} \left[\sum_{i=1}^{2} \int_{S_i} J_i\left(r',\theta'\right) \ln f\left(r,r',\theta,\theta'\right) dSi - \right. \tag{14}$$
$$\left. - I \ln f(r,r_0,\theta,\theta_0)\right] + K_2, (r,\theta) \in S_2$$

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where (r_0, θ_0) specifies the position of the filament, and

$$f\left(r, r', \theta, \theta'\right) = \ln \frac{r^2 + r'^2 - 2rr' \cos\left(\theta - \theta'\right)}{D^2}.$$

In our specific case two conductors are actually the two regions of the inhomogeneous conductor so that $J_1(r,\theta) = J_1^p(r,\theta)$, $J_2(r,\theta) = J_2^p(\theta), dS_1 = r'dr'd\theta', r_0 = \mathrm{D}, \theta_0 = 0$. Since d is very small we may take $dS_2 \approx bdd\theta', r \approx b$ in (14) and $r' \approx b$ in the integral over S_2 in (13). By replacing $J_1(r,\theta)$ and $J_2(r,\theta)$ in (13) - (14) with $J_1^p(r,\theta)$ and $J_2^p(\theta)$ from (11) - (12) we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left[A_n I_n\left(k_1 r\right) + B_n K_n\left(k_1 r\right)\right] \cos n\theta = \\
&= \frac{k_1^2}{4\pi} \left[\sum_{n=1}^{\infty} A_n R_n(r,\theta) + \sum_{n=1}^{\infty} B_n S_n(r,\theta) + \right. \qquad (15) \\
&\left. + bd \sum_{n=1}^{\infty} C_n T_n(r,\theta) - I \ln \frac{r^2 + D^2 - 2rD\cos\theta}{D^2}\right] + K_1 \\
&\sum_{n=1}^{\infty} C_n \cos n\theta = \\
&= \frac{k_1^2}{4\pi} \left[\sum_{n=1}^{\infty} A_n U_n(r,\theta) + \sum_{n=1}^{\infty} B_n V_n(r,\theta) + \right. \qquad (16) \\
&\left. + bd \sum_{n=1}^{\infty} C_n W_n(\theta) - I \ln \frac{b^2 + D^2 - 2bD\cos\theta}{D^2}\right] + K_2
\end{aligned}$$

where R_n, S_n, T_n, U_n, V_n and W_n are some integrals given by

$$R_n(r,\theta) = \int_0^{2\pi} \cos n\theta' d\theta' \int_a^b r' I_n\left(k_1 r'\right) \ln \frac{r^2 + r'^2 - 2rr'\cos\left(\theta - \theta'\right)}{D^2} dr' \qquad (17)$$

$$S_n(r,\theta) = \int_0^{2\pi} \cos n\theta' d\theta' \int_a^b r' K_n\left(k_1 r'\right) \ln \frac{r^2 + r'^2 - 2rr'\cos\left(\theta - \theta'\right)}{D^2} dr' \qquad (18)$$

$$T_n(r,\theta) = \int_0^{2\pi} \cos n\theta' d\theta' \ln \frac{r^2 + b^2 - 2rb\cos\left(\theta - \theta'\right)}{D^2} d\theta \qquad (19)$$

$$U_n(r,\theta) = \int_0^{2\pi} \cos n\theta' d\theta' \int_a^b r' I_n\left(k_1 r'\right) \ln \frac{b^2 + r'^2 - 2br'\cos\left(\theta - \theta'\right)}{D^2} dr' \cong R_n(b,\theta) \qquad (20)$$

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$$V_n(r,\theta) = \int_0^{2\pi} \cos n\theta' d\theta' \int_a^b r' K_n\left(k_1 r'\right) \ln \frac{b^2 + r'^2 - 2br' \cos\left(\theta - \theta'\right)}{D^2} dr' \cong S_n(b,\theta) \tag{21}$$

$$W_n(r,\theta) = \int_0^{2\pi} \cos n\theta' \ln \frac{4b^2 \sin^2 \frac{(\theta-\theta')}{2}}{D^2} d\theta \equiv T_n(b,\theta) \tag{22}$$

Integrals (17)-(19) have been evaluated earlier. This is done in [6] for R_n and S_n and in [8] for T_n. The remaining three integrals U_n, V_n and W_n follow then from R_n, S_n and T_n respectively by putting $r = b$. All these results (we will not write them down) should be substituted into (15)-(16).

If we now equate the coefficients with $r^n \cos n\theta$ and the coefficients with $r^{-n} \cos n\theta$ on both sides of (15)-(16), the following two equations follow

$$A_n I_{n-1}\left(k_1 b\right) - B_n K_{n-1}\left(k_1 b\right) + C_n k_1 d = \frac{I k_1}{\pi b} \left(\frac{b}{D}\right)^n \tag{23}$$

$$A_n I_{n+1}\left(k_1 a\right) - B_n K_{n+1}\left(k_1 a\right) = 0 \tag{24}$$

The third equation that is necessary for determining the three unknown coefficients is obtained by equating the coefficients with $\cos n\theta$ on both sides of (16). It is:

$$A_n M_n + B_n N_n + C_n k_1 d \left(1 + \frac{n}{j\lambda_2}\right) = \frac{I k_1}{\pi b} \left(\frac{b}{D}\right)^n \tag{25}$$

where

$$j\lambda_2 = \frac{k_2^2 bd}{2}$$

From (23) - (25) we can find the unknown coefficients

$$A_n = \frac{I k_1^2}{\pi b} \left(\frac{b}{D}\right)^n \frac{K_{n+1}\left(k_1 a\right)}{k_1 P_n + k_2^2 d Q_n} \tag{26}$$

$$B_n = \frac{I k_1^2}{\pi b} \left(\frac{b}{D}\right)^n \frac{I_{n+1}\left(k_1 a\right)}{k_1 P_n + k_2^2 d Q_n} \tag{27}$$

$$C_n = \frac{I k_2^2}{\pi b} \left(\frac{b}{D}\right)^n \frac{Q_n}{k_1 P_n + k_2^2 d Q_n} \tag{28}$$

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where

$$P_n = I_{n-1}(k_1 b) K_{n+1}(k_1 a) - I_{n+1}(k_1 a) K_{n-1}(k_1 b) \tag{29}$$

$$Q_n = I_n(k_1 b) K_{n+1}(k_1 a) - I_{n+1}(k_1 a) K_n(k_1 b) \tag{30}$$

It should be noted that several formulas from the Bessel function theory have to be used in deriving coefficients (26)(28).

A remarkable feature of the integral equation method that we used to determine the unknown coefficients in the proximity effect solution is that no boundary conditions are used, although the conductor is inhomogeneous. Furthermore, it may be shown that (26)-(28) ensure that the appropriate boundary conditions (equality of the tangential components of the electric and magnetic fields at the interface $r = b$) are met.

2.3 Solution for the total current densities

This solution is obtained by summing the skin and proximity-effect solutions

$$J_1(r,\theta) = A_0 I_0(k_1 r) + B_0 K_0(k_1 r) + \\ + \sum_{n=1}^{\infty} [A_n I_n(k_1 r) + B_n K_n(k_1 r)] \cos n\theta \tag{31}$$

$$J_2(\theta) = C_0 + \sum_{n=1}^{\infty} C_n \cos n\theta \tag{32}$$

with $A_n, B_n, C_n, n \geq 0$ given by (8) - (10), (26) - (28).

From the obtained results some special cases considered earlies immediately follow: solid hollow conductor and filament [6] (we let $\sigma_2 = 0\,(k_2 = 0)$ in the expressions for $A_n, B_n, C_n, n \geq 0$) and thin two-layer tubular conductor and filament [7] (we let $a = b - d, d << a$).

3 Power loss in the conductor

The total power loss in the conductor is obtained by summing power losses in the conductor core ($a < \mathrm{r} < b$) and in the thin coat layer. These losses are calculated by Joule's low.

The power loss per unit lenhght in the conductor core is

$$P'_{J_1} = \frac{1}{\sigma_1} \int_{S_1} |J_1|^2 \, r dr d\theta \tag{33}$$

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By using (31) we can write

$$|J_1|^2 = J_1 \cdot J_1^* = \sum_{n=0}^{\infty} D_n(r)\cos n\theta \cdot \sum_{n=0}^{\infty} D_n^*(r)\cos n\theta = \tag{34}$$
$$= \sum_{n=0}^{\infty} |D_n(r)|^2 \cos^2 n\theta + \sum_{\substack{m,n=0 \\ (m\neq n)}}^{\infty} D_m(r) D_n^*(r)\cos m\theta \cos n\theta$$

where $D_n(r) = A_n I_n(k_1 r) + B_n K_n(k_1 r)$.

The last summation in (34) does not contribute to the integral in (33), due to ortogonality of the cosine function, hence

$$P'_{J_1} = \frac{1}{\sigma_1}\int_0^{2\pi} \cos^2 n\theta d\theta \int_a^b r\,|D_n(r)|^2\,dr =$$
$$= \frac{2\pi}{\sigma_1}\left(\int_a^b r\,|D_0(r)|^2\,dr + \frac{1}{2}\sum_{n=1}^{\infty}\int_a^b r\,|D_n(r)|^2\,dr\right). \tag{35}$$

The integrals in (35) can be evaluated in a closed form [2], in terms of the Kelvin functions, or alternatively by numerical techniques.

The power loss per unit length in the thin layer is

$$P'_{J_1} = \frac{bd}{\sigma_1}\int_0^{2\pi} |J_2|^2\,d\theta$$

where

$$|J_2|^2 = J_2 \cdot J_2^* = \sum_{n=0}^{\infty} C_n \cos n\theta \cdot \sum_{n=0}^{\infty} C_n^* \cos n\theta =$$
$$= \sum_{n=0}^{\infty} |C_n|^2 \cos^2 n\theta + \sum_{\substack{m,n=0 \\ (m\neq n)}}^{\infty} C_n C_n^* \cos m\theta \cos n\theta \tag{36}$$

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As explaned above, we may ignore the last summation in (36), hence

$$P'_{J_2} = \frac{2\pi bd}{\sigma_1}\left(|C_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}|C_n|^2\right)$$

4 Conclusion

A rigorous analysis of the current distribution in a massive hollow cylindrical conductor with a thin layer in the presence of a filament is presented in this paper. Closed form solutions are found in the two regions in the form of infinite sums of the proper harmonics, and the unknown coefficients are found from boundary conditions in the skin-effect case, and from two integral equations in the proximity-effect case. It is remarkable that no boundary conditions are required in the latter case. Formulas for the power loss in the conductor are also included.

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