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WOLD DECOMPOSITION FOR OPERATORS CLOSE ISOMETRIES

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Abstract

The Wold decomposition theorem is used to decompose a wide sense stationary random process into its deterministic and purely nondeterministic parts and, also, has an important role in signal processing, factorization theory and the description of dilations in Hilbert spaces. In geometric language, this means that an isometry is a direct sum between an unitary operator and a shift.

In Bergman and Dirichlet spaces, the shift operator is not an isometry, but it is a left invertible operator. In this paper we give conditions on the left invertible operators such that a operator version, in the sense of Rosenblum and Rovnyak, of the Wold decomposition to take place.[1]

1 Introduction

The Wold decomposition theorem [18] applies to the analysis of stationary random processes. It provides a representation of such processes and also an interpretation of the representation in terms of linear prediction: an arbitrary unpredictable process can be written as an orthogonal sum of a regular process and a predictable process [18].

In 1961, Paul R. Halmos [9] gave the following form of the Wold decomposition theorem in operator language:

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Theorem 1.1. *Let V be an isometry on a Hilbert space $\mathcal{H}$. Then there is a decomposition of $\mathcal{H}$ as a direct sum of two mutually orthogonal subspaces*

$$\mathcal{H} = \mathcal{H}_\infty \bigoplus \mathcal{H}_s$$

such that

(i) $\mathcal{H}_\infty$ and $\mathcal{H}_s$ reduce V.

(ii) The restriction of V to $\mathcal{H}_\infty$ is an unitary operator.

(iii) The restriction of V to $\mathcal{H}_s$ is unitarily equivalent to an unilateral shift.

The decomposition is unique.

We recall that a subspace $\mathcal{H}_0$ of $\mathcal{H}$ reduce V if $\mathcal{H}_0$ is invariant to V and its adjoint. In fact, the subspaces $\mathcal{H}_\infty$ and $\mathcal{H}_s$ are obtained in the following manner

$$\mathcal{H}_\infty = \bigcap_{n=1}^{\infty} V^n \mathcal{H}$$

$$\mathcal{H}_s = \bigoplus_{n=1}^{\infty} V^n W,$$

where $W := \mathcal{H} \ominus V\mathcal{H}$ is the orthogonal complement of $V\mathcal{H}$ in $\mathcal{H}$. Here, W is a wandering subspace for V, that is

$$V^m W \perp V^n W, \quad m \neq n.$$

See also [10], [14].

Theorem 1.1 has some remarkable consequences (see [3], [11]) such as:

- the deduction of the Beurling's invariant subspace Theorem in Hardy spaces;
- the description of the structure of a wide-sense stationary random sequence;
- the description of the structure of isometric and unitary dilation spaces for contractions of a Hilbert space (see Nagy and Foiaş [17]).

The above version of the Wold decomposition emphasizes spatial structure. An operator version of the Wold decomposition of an isometry is given by M. Rosenblum and J. Rovyak in their book [13].

But in Bergman and Dirichlet spaces the shift operator is no longer an isometry. Remarkable Wold type Theorems for classes of left invertible operators and applications to problems of invariant subspaces were obtained by S. Richter [12]

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and S. Shimorin [16]. In 1991, A. Aleman, S. Richter, C. Sundberg [2] proved the Beurling type theorem for Bergman shift, which was a big step in the study of invariant subspaces of the Bergman shift. This result became an important tool in the function theory of L^2_a because it shows the structure of invariant subspaces of the Bergman space.

This paper is motivated by a problem posed by S. Shimorin [16]. The problem is to give new conditions for a left invertible operator to imply Wold type decompositions. The aim of this paper is to give conditions on the left invertible operators such that a operator version of the Wold decomposition can be proved. The left invertible operators (the operators bounded below) are the analysis operators from frame theory (see [5], [8]).

2 Results

We denote by $\mathcal{L}(\mathcal{H})$ the algebra of all linear bounded operators on the Hilbert space $\mathcal{H}$ and for $T \in \mathcal{L}(\mathcal{H})$, we denote by T^* the adjoint operator of T. The following lemma is a well-known result.

Lemma 2.1. *Let be $T \in \mathcal{L}(\mathcal{H})$. The following are equivalent*

(i) T is left invertible;

(ii) T is bounded below, i.e. there exists a constant $m > 0$ such that

$$\|Th\| \geq m\|h\|, \quad h \in \mathcal{H};$$

(iii) T^* is surjective;

(iv) T^*T is invertible.

If T is left invertible, then $T\mathcal{H}$ is a closed subspace of $\mathcal{H}$. As in [12, 16] we distinguish the following left-inverse of T

$$T^- = (T^*T)^{-1}T^*$$

and its kernel

$$W = \mathcal{H} \ominus T\mathcal{H} = KerT^*.$$

The subspace W is called the defect of T. It is clear that if T is left invertible then T^n is left invertible.

In the following, $\mathcal{D}$ denotes the set of all left invertible operators on $\mathcal{H}$ for which the following condition holds

$$(T^n)^- = (T^-)^n, \quad \text{for all } n \geq 2. \tag{1}$$

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Remark 2.2. *If T is an isometry, then $T^*T = I$, hence $T^- = T^*$ and (1) holds. In fact, if T is left invertible and $T^- = T^*$, then T is an isometry. Indeed, from the relation $(T^*T)^{-1}T^* = T^*$ it follows*

$$(T^*T)^{-1}T^*T = T^*T$$

hence $I = T^*T$.

In the following we give conditions for operators to be in the class $\mathcal{D}$.

Proposition 2.3. *Let $Q \in \mathcal{L}(\mathcal{H})$ be left invertible and quasinormal. Then $Q \in \mathcal{D}$.*

Proof. Since Q quasinormal, we have $(Q^*Q)Q = Q(Q^*Q)$.

We claim that this implies that $Q^{*n}Q^n = (Q^*Q)^n$, for all $n \geq 2$. We prove this by induction on n.

Indeed, for $n = 2$

$$Q^{*2}Q^2 = Q^*(Q^*Q)Q = (Q^*Q)(Q^*Q)$$

If $Q^{*n}Q^n = (Q^*Q)^n$ then

$$\begin{aligned}(Q^*Q)^{n+1} &= Q^*Q(Q^*Q)^n = (Q^*Q)Q^{*n}Q^n \\ &= Q^{*n}(Q^*Q)Q^n.\end{aligned}$$

We used the fact that Q is quasinormal, hence $(Q^*Q)Q^* = Q^*(Q^*Q)$ and by induction, we have $(Q^*Q)Q^{*n} = Q^{*n}(Q^*Q)$.

So we get$(Q^*Q)^{n+1} = Q^{*n+1}Q^{n+1}$.

It follows

$$(Q^*Q)^{-n} = (Q^{*n}Q^n)^{-1}.$$

Hence

$$\begin{aligned}(Q^-)^n &= [(Q^*Q)^{-1}Q^*]^n = Q^{*n}(Q^*Q)^{-n} \\ &= Q^{*n}(Q^{*n}Q^n)^{-1} = (Q^n)^-.\end{aligned}$$

□

We recall that two operators T_1, T_2 are *double commuting* if $T_1T_2{=}T_2T_1$ and $T_1T_2^*{=}T_2^*T_1$

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Proposition 2.4. *Let T_1, T_2 be double commuting operators in $\mathcal{D}$. Then*

$$T_1T_2 \in \mathcal{D}.$$

Proof. It is clear that T_1T_2 is left invertible since

$$T_2^-T_1^-T_1T_2 = T_2^-T_2 = I.$$

From hypothesis,

$$T_2(T_1^*T_1) = (T_1^*T_1)T_2$$

hence

$$(T_1^*T_1)^{-1}T_2 = T_2(T_1^*T_1)^{-1}.$$

It follows that

$$\begin{aligned} T_1T_2 &= [(T_1T_2)^*(T_1T_2)]^{-1}(T_1T_2)^* \\ &= (T_2^*T_2T_1^*T_1)^{-1}T_2^*T_1^* \\ &= (T_1^*T_1)^{-1}(T_2^*T_2)^{-1}T_1^*T_2^* \\ &= (T_1^*T_1)^{-1}T_1^*(T_2^*T_2)^{-1}T_2^* \\ &= T_1^-T_2^-. \end{aligned}$$

From here we get

$$\begin{aligned} [(T_1T_2)^-]^n &= (T_1^-T_2^-)^n = (T_1^-)^n(T_2^-)^n \\ &= (T_1^n)^-(T_2^n)^- = (T_1^nT_2^n)^- = [(T_1T_2)^n]^-, \end{aligned}$$

since $T_1^n, T_2^n \in \mathcal{D}$ and T_1^n and T_2^n are double commuting. □

Corollary 2.5. *Let be $T_1 \in \mathcal{D}$ and T_2 be normal and invertible, and $T_1T_2 = T_2T_1$. Then $T_1T_2 \in \mathcal{D}$.*

Proof. From the hypothesis it follows that $T_2 \in \mathcal{D}$. From the Fuglede-Putnam theorem [10] it follows $T_1^*T_2 = T_2T_1^*$. Now, the conclusion is a consequence of Proposition 2.4. □

Remark 2.6. *From the above corrollary, we also obtain that every quasinormal and left invertible operator is in $\mathcal{D}$.*

Indeed, T admits polar decomposition $T = VA$, with V isometry, $A = (T^*T)^{1/2}$ and $VA = AV$ (see [7]).

Next, we give an example of quasinormal, left invertible operator that is not an isometry.

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Let $\mathcal{K}$ be a Hilbert space of dimension at least 2 and

$$l^2(\mathcal{K}) = \{\widetilde{k} = (k_0, k_1, \ldots, k_n, \ldots) | k_j \in \mathcal{K}, j = 0, 1, \ldots\} \text{ and } \|\widetilde{k}\|_2^2 := \sum_{j=0}^{\infty} \|k_j\|^2 < \infty\}.$$

Let $L \in \mathcal{L}(\mathcal{K})$ be a positive invertible operator such that $\|Lk\| \geq m\|k\|$, for all k and some $m > 1$. We define the following operator on $l^2(\mathcal{K})$:

$$T\widetilde{k} := (0, Lk_0, Lk_1, \ldots)$$

Note that T is bounded on $l^2(\mathcal{K})$ and is not surjective;

$$(T^*T\widetilde{k})_n = L^2 k_n$$

$$[T(T^*T)\widetilde{k}] = (0, L^3 k_0, \ldots, L^3 k_n, \ldots)$$

$$(T^*T)T\widetilde{k} = T^*T(0, Lk_0, \ldots, Lk_n, \ldots) = (0, L^3 k_0, \ldots, L^3 k_n, \ldots)$$

It follows that T is quasinormal.
We have $\|T\widetilde{k}\|_2^2 = \sum \|Lk_n\|^2 \geq m^2 \sum \|k_n\|^2$.
This implies that T is not an isometry, since $\|T\widetilde{k}\|_2^2 \neq \sum \|k_n\|^2$.

We give conditions for the weighted shifts [15] and weighted translation operators [6] to be in $\mathcal{D}$.

Proposition 2.7. *Every bounded left invertible unilateral weighted shift on l^2 is in the class $\mathcal{D}$.*

Proof. Let T be a unilateral weighted shift, which is bounded below, ie
$Te_k = w_k e_{k+1}, \quad k \geq 0$ and $C_1 \leq w_n \leq C_2$, where C_1, C_2 are positive constants.
We have:

$$T^* e_k = \begin{cases} \overline{w}_{k-1} e_{k-1}, & k \geq 1 \\ 0, & k = 0; \end{cases}$$

$$T^n e_k = w_k w_{k+1} \cdots w_{k+n-1} e_{k+n}, \text{ and}$$

$$T^{*n} e_k = \begin{cases} \overline{w}_{k-1} \overline{w}_{k-2} \cdots \overline{w}_{k-n} e_{k-n}, & k \geq n \\ 0, & 0 \leq k < n. \end{cases}$$

It follows

$$\begin{aligned} (T^-)^* e_k = T(T^*T)^{-1} e_k &= T\left(\frac{1}{|w_k|^2} e_k\right) \\ &= \frac{1}{|w_k|^2} w_k e_{k+1} \\ &= \frac{1}{\overline{w}_k} e_{k+1} \end{aligned}$$

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and

$$(T^-)^{*n}e_k = \frac{1}{\overline{w}_k\overline{w}_{k+1}\cdots\overline{w}_{k+n-1}}e_{k+n}.$$

On the other hand,

$$T^{*n}T^n e_k = w_k w_{k+1}\cdots w_{k+n-1}\overline{w}_{k+n-1}\cdots\overline{w}_k e_k$$

which implies

$$\begin{aligned}[(T^n)^-]^* &= T^n\left(\frac{1}{|w_k|^2\cdots|w_{k+n-1}|^2}\right)e_k \\ &= \frac{1}{\overline{w}_k\cdots\overline{w}_{k+n-1}}e_{k+n}\end{aligned}$$

It follows $(T^-)^n = (T^n)^-,$ for all $n \geq 2$. □

From the above Proposition, it follows that the Bergman shifts, i.e. the shifts with sequence weights $\left\{\sqrt{\frac{k+1}{k+2}}\right\}_{k\in\mathbb{N}}$ and also, the Dirichlet shifts, i.e. the shifts with sequence weights $\left\{\sqrt{\frac{k+2}{k+1}}\right\}_{k\in\mathbb{N}}$ are in the class of $\mathcal{D}$.

Proposition 2.8. *Every left invertible weighted translation operator on $L^2(0,\infty)$ is in $\mathcal{D}$.*

Proof. Let T be a weighted translation operator, i.e.

$$Tf(x) = \begin{cases}\dfrac{\varphi(x)}{\varphi(x-t)}f(x-t), \text{ if } x > t \\ 0, \text{ if } 0 < x \leq t\end{cases}$$

Further we have

$$T^*f(x) = \frac{\varphi(x+t)}{\varphi(x)}f(x+t)$$

$$T^nf(x) = \frac{\varphi(x)}{\varphi(x-nt)}f(x-nt), \text{ for } x > nt$$

and

$$T^{*n}f(x) = \frac{\varphi(x+nt)}{\varphi(x)}f(x+nt).$$

It follows that

$$T^{*n}T^nf(x) = \left[\frac{\varphi(x+nt)}{\varphi(x)}\right]^2 f(x),$$

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$$T^*Tf(x) = \left[\frac{\varphi(x+t)}{\varphi(x)}\right]^2 f(x),$$

$$T^-f(x) = \frac{\varphi(x)}{\varphi(x+t)} f(x+t),$$

$$(T^-)^n f(x) = \frac{\varphi(x)}{\varphi(x+nt)} f(x+nt) \text{ and}$$

$$(T^n)^- f(x) = \frac{\varphi(x)}{\varphi(x+nt)} f(x+nt).$$

□

Next, we give the main result of this paper. We recall here the following notation $T^- = (T^*T)^{-1}T^*$.

Theorem 2.9. *Let $T \in \mathcal{L}(\mathcal{H})$ be in $\mathcal{D}$. Then*

(i) $P_0 := I - TT^-$ *is the projection of* $\mathcal{H}$ *on* $W := \mathcal{H} \ominus T\mathcal{H}$;

(ii) as $n \to \infty$, $T^n(T^-)^n$ *converges strongly to the projection operator,* P, *on*

$$\bigcap_{n=1}^{\infty} T^n\mathcal{H};$$

(iii) $\sum_{j=0}^{\infty} T^j P_0 (T^-)^j$ *converges strongly to* $Q := I - P$;

(iv) $Q\mathcal{H} = \{h \in \mathcal{H} : \lim_{n\to\infty} \|(T^{*n}T^n)^{-1/2}T^{*n}\| = 0\}$;

(v) $P\mathcal{H}$ *and* $Q\mathcal{H}$ *reduce* T;

(vi) $T|_{P\mathcal{H}}$ *is surjective;*

(vii) $I = P + \sum_{j=0}^{\infty} T^j P_0 (T^-)^j$

Proof. Let $P_n = T^n(T^-)^n$, $n \geq 1$. We prove that P_n is the orthogonal projection of $\mathcal{H}$ on $T^n\mathcal{H}$, $n \geq 1$. Indeed,

$$P_1^2 = T(T^-T)T^- = TT^- = P_1$$

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$$P_1^* = (T^-)^* T^* = T(T^*T)^{-1}T^* = TT^- = P_1.$$

Hence P_1 is the orthogonal projection of $\mathcal{H}$ on $P_1\mathcal{H} = TT^-\mathcal{H} = T\mathcal{H}$ since T^- is surjective. For $n \geq 2$, T^n is also left invertible.

From the above result it follows that $P_n = T^n(T^n)^- = T^n(T^-)^n$ is the orthogonal projection of $\mathcal{H}$ on $T^n\mathcal{H}$.

It is clear that $P_0 : I - TT^-$ is the orthogonal projection of $\mathcal{H}$ on $\mathcal{H} \ominus T\mathcal{H}$.

We prove that $P_n h \to Ph$, for all $h \in \mathcal{H}$ and $T^m W \perp T^n W, \quad m \neq n$.
It is clear that $P_n - P_{n+1}$ is the orthogonal projection of $\mathcal{H}$ on $T^n\mathcal{H} \bigcap (T^{n+1}\mathcal{H})^\perp$.

It follows

$$(P_n - P_{n+1})\mathcal{H} \perp (P_m - P_{m+1})\mathcal{H}, m \neq n.$$

Hence

$$\begin{aligned}\sum_{n=0}^{m} \|P_n h - P_{n+1}h\|^2 &= \|\sum_{n=0}^{m}(P_n h - P_{n+1}h)\|^2 \\ &= \|h - P_{m+1}h\|^2 \leq 4\|h\|^2\end{aligned}$$

It follows that $\sum_{n=0}^{\infty} \|P_n h - P_{n+1}h\|^2$ converges, i.e. for every $\varepsilon > 0$, there exists an $N(\varepsilon)$ such that for $n \geq N(\varepsilon)$, we have:

$$\begin{aligned}&\|P_n h - P_{n+1}h\|^2 + \|P_{n+1}h - P_{n+2}h\|^2 + \ldots + \|P_{n+p-1}h - P_{n+p}h\|^2 < \varepsilon \\ &\iff \|P_n h - P_{n+p}h\|^2 < \varepsilon, \text{ for all } p \in \mathbb{N}, p \geq 1.\end{aligned}$$

So $(P_n h)$ converges to an element in $\mathcal{H}$.

We denote $Ph = \lim_{n\to\infty} P_n h$. We prove that P is the orthogonal projection of $\mathcal{H}$ on $\bigcap_{n=1}^{\infty} T^n\mathcal{H}$.

Indeed, we consider $h \in \bigcap_{n=1}^{\infty} T^n\mathcal{H}$. Then $h \in T^n\mathcal{H}$, for all $n \geq 1$ and $P_n h = h$, for all $n \geq 1$. Hence $Ph = h$.

On the other hand, if we take $h \perp \bigcap_{n=1}^{\infty} T^n\mathcal{H}$. Notice that, by the definition of P, it follows that $Ph \in \bigcap_{n=1}^{\infty} T^n\mathcal{H}$. Then

$$P(Ph) = \lim_{n\to\infty} P_n(Ph) = \lim_{n\to\infty} Ph = Ph$$

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and hence

$$\|Ph\|^2 = \langle Ph, Ph\rangle = \langle P^2 h, h\rangle = \langle Ph, h\rangle = 0.$$

Thus $Ph = 0$.

We have

$$\begin{aligned} P_n - P_{n+1} &= T^n (T^-)^n - T^{n+1}(T^-)^{n+1} \\ &= T^n (I - TT^-)(T^-)^n \\ &= T^n P_0 (T^-)^n \end{aligned}$$

and

$$(I - P_1) + (P_1 - P_2) + \ldots + (P_n - P_{n+1}) = I - P_{n+1}.$$

Hence $\sum_{n=0}^{\infty} T^n P_0 (T^-)^n$ converges to $I - P = Q$.

For proving (iv), we observe that

$$h \in Q\mathcal{H} \Longleftrightarrow Ph = 0 \Longleftrightarrow \lim_{n\to\infty} \|T^n (T^-)^n h\| = 0.$$

The last equality is equivalent with

$$\lim_{n\to\infty} \langle T^n (T^{*n} T^n)^{-1} T^{*n} h, T^n (T^{*n} T^n)^{-1} T^{*n} h\rangle = 0$$

i.e. $\lim_{n\to\infty} \|(T^{*n} T^n)^{-1/2} T^{*n} h\| = 0$.

For proving that $\mathcal{H}_\infty$ reduces T we note that

$$\begin{aligned} P_{n+1} T &= T^{n+1}(T^-)^{n+1} T \\ &= T^{n+1}(T^-)^n (T^- T) = T^{n+1}(T^-)^n \\ TP_n &= TT^n (T^-)^n = T^{n+1}(T^-)^n \end{aligned}$$

Hence $P_{n+1}T = TP_n \Rightarrow PT = TP$ hence $\mathcal{H}_\infty$ reduces T.

Next we prove now that $T|_{\mathcal{H}_\infty}$ is surjective.

Let $h_0 \in \mathcal{H}_\infty$. It follows $h_0 \in T^n \mathcal{H},$ for all $n \geq 1$.

For any $n \geq 1$ there exists $h_n \in \mathcal{H}$ so that $h_0 = T^n h_n$. Then

$$h_0 = Th'_n, h'_n \in T^{n-1}\mathcal{H}, n \geq 1 \Rightarrow T^- h_0 = h'_n, n \geq 1 \Rightarrow h'_n = h'_1, n \geq 1.$$

Hence

$$h_0 = Th'_1, h'_1 \in T^{n-1}\mathcal{H}, n \geq 1 \Rightarrow h'_1 \in \bigcap_{n\geq 1} T^{n-1}\mathcal{H} = \mathcal{H}_\infty.$$

□

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Theorem 2.10. *Let $T \in \mathcal{D}$. Then $W := \mathcal{H} \ominus T\mathcal{H}$ is a wandering subspace of $\mathcal{H}$ and*

$$\mathcal{H} = \mathcal{H}_\infty \oplus \mathcal{H}_s,$$

where

$$\mathcal{H}_\infty = \bigcap_{n=1}^{\infty} T^n\mathcal{H}, \quad \mathcal{H}_s = \bigoplus_{n=0}^{\infty} T^n W.$$

$\mathcal{H}_\infty$ and $\mathcal{H}_s$ are reducing spaces of T and $T|_{\mathcal{H}_\infty}$ is bijective. The decomposition is unique.

Proof. The fact that the decomposition exists follows from Theorem 2.9. We prove that the decomposition is unique. Let $\mathcal{H} = \mathcal{H}'_\infty \bigoplus \mathcal{H}'_s$ a decomposition such that

$$T\mathcal{H}'_\infty = \mathcal{H}'_\infty$$

$$\mathcal{H}'_s = \bigoplus_{n=0}^{\infty} T^n W',$$

where W' is a wandering subspace of T. We prove that $\mathcal{H}'_\infty = \mathcal{H}_\infty$ and $\mathcal{H}'_s = \mathcal{H}_s$. Indeed,

$$\begin{aligned} W = \mathcal{H} \ominus T\mathcal{H} &= (\mathcal{H}'_\infty \bigoplus \mathcal{H}'_s) \ominus (T\mathcal{H}'_\infty \bigoplus T\mathcal{H}'_s) \\ &= (\mathcal{H}'_\infty \bigoplus \mathcal{H}'_s) \ominus (\mathcal{H}'_\infty \bigoplus T\mathcal{H}'_s) \\ &= \mathcal{H}'_s \ominus T\mathcal{H}'_s = W' \end{aligned}$$

We use the following fact:
If $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is such that $\mathcal{H}_1$, $\mathcal{H}_2$ are reducing subspaces of T, then

$$T\mathcal{H} = T\mathcal{H}_1 \oplus T\mathcal{H}_2.$$

This is clear because

$$h = h_1 + h_2, \quad h_1 \perp h_2, \quad h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2$$

$Th = Th_1 + Th_2$ and $\langle Th_1, Th_2\rangle = \langle T^*Th_1, h_2\rangle = 0$. □

The following result is a Wold-type decomposition for a pair of double commuting operators in $\mathcal{D}$.

Theorem 2.11. *Let be $T_1, T_2 \in \mathcal{D}$ double commuting. Then $\mathcal{H}$ has the following orthogonal decomposition*

$$\mathcal{H} = \mathcal{H}_{\infty\infty} \bigoplus \mathcal{H}_{\infty s} \bigoplus \mathcal{H}_{s\infty} \bigoplus \mathcal{H}_{ss},$$

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where $\mathcal{H}_{\infty\infty}$, $\mathcal{H}_{\infty s}$, $\mathcal{H}_{s\infty}$, $\mathcal{H}_{ss}$ *are reducing spaces of* $T_i (i = 1, 2)$ *and*

$$\mathcal{H}_{\infty\infty} = \bigcap_{m,n=1}^{\infty} T_1^m T_2^n \mathcal{H}$$

$$\mathcal{H}_{\infty s} = \Big(\bigcap_{m=1}^{\infty} T_1^m \mathcal{H} \Big) \bigcap \Big(\bigoplus_{n=0}^{\infty} T_2^n W_2 \Big)$$

$$\mathcal{H}_{s\infty} = \Big(\bigoplus_{m=0}^{\infty} T_1^m W_1 \Big) \bigcap \Big(\bigcap_{n=1}^{\infty} T_2^n \mathcal{H} \Big)$$

$$\mathcal{H}_{ss} = \Big(\bigoplus_{m=0}^{\infty} T_1^m W_1 \Big) \bigcap \Big(\bigoplus_{n=0}^{\infty} T_2^n W_2 \Big).$$

Proof. We denote by Q_i the orthogonal projection on

$$\bigcap_{n=1}^{\infty} T_i^n \mathcal{H}, \ (i = 1, 2)$$

From hypothesis, Q_1, Q_2 are commuting. The decomposition given in Theorem 2.11 follows from the identity

$$I = Q_1 Q_2 + Q_1 (I - Q_2) + (I - Q_1) Q_2 + (I - Q_1)(I - Q_2).$$

□

Comments. The results of this paper appeared on Arxiv, in 2017, under my maiden name, Laura Găvruţa: https://arxiv.org/abs/1704.04200.

After the paper was posted on Arxiv, we saw the paper [4] and the book [1], in connection with the subject of our paper.

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