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SPECTRAL MAPPING THEOREM FOR GENERALIZED EVOLUTION SEMIGROUPS

Nicolae LUPA

Abstract

In the paper [N. Lupa, L. H. Popescu, *Generalized evolution semigroups and general dichotomies*, Results Math. 78 (2023): 112] the authors introduced a special class of real semiflows in order to associate evolution semigroups to not necessarily exponentially bounded evolution families, called as *generalized evolution semigroups*. We give a direct proof of the spectral mapping theorem for generalized evolution semigroups. [5]

1 Introduction

The theory of evolution semigroups has a significant importance in the study of the asymptotic behavior of their underlying evolution families. For instance, the exponential dichotomy of an evolution family is equivalent to the hyperbolicity of its corresponding evolution semigroup (see [4, Theorem VI.9.18]). The key to the proof lies in the spectral mapping theorem: *the evolution semigroup* $\{T_t\}_{t\geq 0}$ *corresponding to an exponentially bounded evolution family satisfies the spectral mapping formula*

$$\sigma(T_t)\setminus\{0\} = e^{t\sigma(G)},\ t\geq 0,$$

where G *is the generator of the semigroup* $\{T_t\}_{t\geq 0}$. However, the classical theory of evolution semigroups is too restrictive since it is only addresses to *exponentially bounded* evolution families.

In [5] the authors successfully constructed a framework for a generalization of the concept of evolution semigroups associated to not necessarily exponentially bounded evolution families in order to characterize a wide class of dichotomies, which can occur naturally for instance when all Lyapunov exponents are infinite or they are all zero [2].

More precisely, it is proved that any non-degenerate real semiflow $\varphi : \mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ is determined by a continuous strictly increasing function $\mu:\mathbb{R}\to\mathbb{R}$ such that $\mu(s)\to-\infty$ as $s\to-\infty$ and

$$\varphi(t,s) = \mu^{-1}(\mu(s)-t),\ \text{for all } t\geq 0 \text{ and } s\in\mathbb{R}$$

[5, Theorem 2.5]. We recall that a *non-degenerate real semiflow* is a continuous function $\varphi : \mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ satisfying the following properties:

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- $\varphi(0,s)=s,\ s\in\mathbb{R}$;
- $\varphi(t,\varphi(\tau,s))=\varphi(t+\tau,s),\ t,\tau\geq 0,\ s\in\mathbb{R}$;
- $\varphi(t,s)\leq s,\ t\geq 0,\ s\in\mathbb{R}$;
- any orbit $o(s)=\{\varphi(t,s):\ t\geq 0\}$ is non-trivial, i.e. $o(s)\neq\{s\}$.

Furthermore, it is proved that any non-degenerate real semiflow $\varphi:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ with

$$\lim_{s\to\infty}\varphi(t,s)=+\infty,\ \text{for } t\geq 0,$$

that is $\mu(s)\to\infty$ as $s\to\infty$, and any evolution family $\{U(t,\tau)\}_{t\geq\tau}$ satisfying the inequality

$$\|U(s,\varphi_t(s))\|\leq Ke^{\alpha t},\ t\geq 0,\ s\in\mathbb{R},$$

for some constants $\alpha>0$ and $K\geq 1$, which is equivalent to

$$\|U(t,\tau)\|\leq Ke^{\alpha(\mu(t)-\mu(\tau))},\ t\geq\tau, \tag{1}$$

define a strongly continuous semigroup $\{T_t\}_{t\geq 0}$ on $C_0(\mathbb{R},E)$ by

$$T_tu(s)=U(s,\varphi_t(s))u(\varphi_t(s))=U(s,\mu^{-1}(\mu(s)-t))u(\mu^{-1}(\mu(s)-t)), \tag{2}$$

called as *generalized evolution semigroup* (see [5, Theorem 3.3]). Remark that if we consider in (2) the right translation semiflow $\varphi(t,s)=s-t$, which means that $\mu(s)=s$, we step over the classical concept of the evolution semigroup,

$$T_tu(s)=U(s,s-t)u(s-t),\ t\geq 0,\ u\in C_0(\mathbb{R},E),\ s\in\mathbb{R}.$$

Using the fact that the generalized evolution semigroup is similar to the classical evolution semigroup (see Sect. 3.2 in [5]) one can easily established the spectral mapping theorem in this context.

The aim of this note is to give a direct proof of the spectral mapping theorem for the generalized evolution semigroup introduced in [5], without using the similarity relation mentioned above.

2 Generalized evolution semigroups

Let E be a complex Banach space and let $\mathcal{B}(E)$ be the Banach algebra of all bounded linear operators on E. To simplify notations both norms on E and on $\mathcal{B}(E)$ are denoted by $\|\cdot\|$. Moreover, $C(\mathbb{R},E)$ is the space of all continuous functions $u:\mathbb{R}\to E$ and $C_0(\mathbb{R},E)$ denotes the Banach space of all functions in $C(\mathbb{R},E)$ vanishing at $\pm\infty$, that is

$$C_0(\mathbb{R},E)=\left\{u\in C(\mathbb{R},E):\ \lim_{|s|\to\infty}u(s)=0\right\},$$

endowed with the sup-norm

$$\|u\|_\infty=\sup_{s\in\mathbb{R}}\|u(s)\|.$$

It is well-known that $C_c(\mathbb{R},E)=\{u\in C(\mathbb{R},E):\ supp(u)\text{ is compact}\}$ is dense in $C_0(\mathbb{R},E)$.

A collection $\mathcal{U}=\{U(t,\tau)\}_{t\geq\tau}$ of bounded linear operators $U(t,\tau)$ on E is called an *evolution family* (on the real line) on E if:

- $U(t,t)=\mathrm{I}$, $t\in\mathbb{R}$, here I denotes the identity on E;
- $U(t,\tau)U(\tau,t_0)=U(t,t_0)$, $t\geq\tau\geq t_0$ in $\mathbb{R}$;
- for each $x\in E$, the mapping $(t,\tau)\mapsto U(t,\tau)x$ is continuous on

$$\Delta=\left\{(t,\tau)\in\mathbb{R}^2:\ t\geq\tau\right\}.$$

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It is known that if $\mathcal{U}$ is *exponentially bounded*, i.e. there exist two real constants $\alpha > 0$ and $K \geq 1$ such that

$$\|U(t,\tau)\| \leq Ke^{\alpha(t-\tau)},\ t \geq \tau, \tag{3}$$

then one can define a strongly continuous semigroup on $X = C_0(\mathbb{R}, E)$ by

$$T_t u(s) = U(s, s-t)u(s-t),\ t \geq 0,\ u \in C_0(\mathbb{R}, E),\ s \in \mathbb{R},$$

called *the evolution semigroup* associated to $\mathcal{U}$ (see, for instance, [4, Lemma VI.9.10] or [3, Proposition 3.11]). The above concept of evolution semigroup was extended by Räbiger and Schnaubelt to a large class of E-valued function spaces, including in particular the space $C_0(\mathbb{R}, E)$ [6].

We recall that a *strongly continuous semigroup* or a C_0*-semigroup* on a Banach space X is a family $\{T_t\}_{t\geq 0}$ of operators in $\mathcal{B}(X)$ satisfying:

- $T_0 = \mathrm{I}$;
- $T_t\, T_\tau = T_{t+\tau},$ for $t, \tau \geq 0$;
- $T_t x \to x$ in X as $t \to 0^+$ for every fixed $x \in X$.

The (infinitesimal) *generator* of a strongly continuous semigroup $\{T_t\}_{t\geq 0}$ is the linear operator G, with domain $D(G)$, defined by

$$D(G) = \left\{ x \in X : \lim_{t\to 0^+} \frac{T_t x - x}{t} \text{ exists } \right\}$$

and

$$Gx = \lim_{t\to 0^+} \frac{T_t x - x}{t},\ x \in D(G).$$

We refer the reader to the monograph of Engel and Nagel [4] for a brief history of strongly continuous semigroups.

Let $\mu : \mathbb{R} \to \mathbb{R}$ be a continuous strictly increasing function such that

$$\lim_{s\to-\infty} \mu(s) = -\infty \text{ and } \lim_{s\to+\infty} \mu(s) = +\infty,$$

and let $\mathcal{U}$ be an evolution family satisfying (1), which is a natural generalization of the condition (3).

From [5, Theorem 3.3] we get that the family $\{T_t\}_{t\geq 0}$ defined by Eq. (2) is a C_0-semigroup on $C_0(\mathbb{R}, E)$, called the *generalized evolution semigroup* associated to μ and $\mathcal{U}$. In the following, we denote by G the generator of the generalized evolution semigroup.

[Rescaled generalized evolution semigroup] It is easy to check that for any $\lambda \in \mathbb{C}$ the evolution family

$$V(t,s) = e^{-\lambda(\mu(t)-\mu(s))}U(t,s),\ t \geq s,$$

satisfies the inequality

$$\|V(t,s)\| \leq Ke^{(\alpha+|\lambda|)(\mu(t)-\mu(s))},\ t \geq s,$$

and thus one can define its generalized evolution semigroup $\left\{T_t^\lambda\right\}_{t\geq 0}$ on $C_0(\mathbb{R}, E)$ with respect to μ. More precisely, T_t^λ is given by

$$T_t^\lambda = e^{-\lambda t}T_t,\ t \geq 0,$$

where $\{T_t\}_{t\geq 0}$ is the generalized evolution semigroup associated to μ and $\mathcal{U}$.

Furthermore, the generator of $\left\{T_t^\lambda\right\}_{t\geq 0}$ is $G - \lambda\mathrm{I}$ and its spectrum is given by

$$\sigma(G - \lambda\mathrm{I}) = \sigma(G) - \lambda.$$

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3 Spectral mapping theorem

We are now able to generalize the spectral mapping theorem for generalized evolution semigroups. Let us first recall some basic notions and results from the spectral theory.

Let $(A, D(A))$ be a closed operator on the Banach space E. A complex number $\lambda \in \mathbb{C}$ belongs to *the residual spectrum* $R\sigma(A)$ if the set $(\lambda \mathrm{I} - A)D(A)$ is not dense in E and belongs to *the approximate point spectrum* $A\sigma(A)$ if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $D(A)$ with $\|x_n\| \geq 1$ such that $(\lambda \mathrm{I} - A)x_n \to 0$.

It is well-known that any C_0-semigroup $\{T_t\}_{t\geq 0}$ with the generator $(A, D(A))$ satisfies the spectral inclusion relation

$$e^{t\sigma(A)} \subset \sigma(T_t),\ t \geq 0. \tag{4}$$

Furthermore, the residual spectra of the semigroup and its generator satisfy

$$e^{tR\sigma(A)} = R\sigma(T_t) \setminus \{0\},\ t \geq 0. \tag{5}$$

Theorem 3.1 (Spectral mapping theorem)**.** *The generalized evolution semigroup $\{T_t\}_{t\geq 0}$ satisfies the spectral mapping formula*

$$\sigma\left(T_t\right) \setminus \{0\} = e^{t\sigma(G)},\ t \geq 0. \tag{6}$$

Proof. We adjust the arguments in [6, Theorem 2.3] for our purpose. Since

$$\sigma(T_t) = R\sigma(T_t) \cup A\sigma(T_t),$$

by (4) and (5) it suffices to prove that

$$A\sigma\left(T_t\right) \setminus \{0\} \subset e^{t\sigma(G)},\ t \geq 0.$$

In fact, this inclusion is a consequence of a weaker assertion:

$$1 \in A\sigma\left(T_{t_0}\right) \text{ for some } t_0 > 0 \text{ implies } 0 \in \sigma(G). \tag{7}$$

Indeed, if $z \in A\sigma\left(T_{t_0}\right) \setminus \{0\}$, then $1 \in A\sigma\left(e^{-\lambda t_0} T_{t_0}\right)$, where $z = e^{\lambda t_0}$. Applying (7) for the C_0-semigroup $\left\{e^{-\lambda t_0} T_{t_0}\right\}$ and using the results in Example 2, we get that $0 \in \sigma(G) - \lambda$. This yields $\lambda \in \sigma(G)$ and thus $z \in e^{t_0\sigma(G)}$.

It remains to prove (7). Assume that $1 \in A\sigma\left(T_{t_0}\right)$ for some $t_0 > 0$. Then there exists a sequence (u_n) in $C_0(\mathbb{R}, E)$ such that $\|u_n\|_\infty = 1$ for all $n \in \mathbb{N}$ and

$$\|T_{t_0} u_n - u_n\|_\infty \to 0.$$

From the inequality

$$\|T_{kt_0} u_n - u_n\|_\infty \leq \left(1 + \|T_{t_0}\| + \cdots + \|T_{t_0}\|^{k-1}\right) \|T_{t_0} u_n - u_n\|_\infty,\ k \in \mathbb{N}^*,$$

we get that for each $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that

$$\|T_{kt_0} u_{k_n} - u_{k_n}\|_\infty < \frac{1}{2},\ \text{for } k = \overline{0, 2n}.$$

Therefore,

$$\frac{1}{2} < \|T_{kt_0} u_{k_n}\|_\infty < \frac{3}{2},\ \text{for } k = \overline{0, 2n},$$

which is equivalent to

$$\frac{1}{2} < \sup_{s\in\mathbb{R}} \|U(s, \mu^{-1}(\mu(s) - kt_0)) u_{k_n}(\mu^{-1}(\mu(s) - kt_0))\| < \frac{3}{2},\ \text{for } k = \overline{0, 2n}. \tag{8}$$

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The left-hand side of the above inequality implies

$$\sup_{s\in\mathbb{R}} \|U(s,\mu^{-1}(\mu(s)-nt_0))u_{k_n}(\mu^{-1}(\mu(s)-nt_0))\| > \frac{1}{2}.$$

For each $n\in\mathbb{N}$ take $s_n\in\mathbb{R}$ such that

$$\|U(s_n,\mu^{-1}(\mu(s_n)-nt_0))x_n\| > \frac{1}{2}, \tag{9}$$

where $x_n = u_{k_n}(\mu^{-1}(\mu(s_n)-nt_0))$.

For each $n\in\mathbb{N}^*$ put $I_n = [\mu(s_n)-nt_0, \mu(s_n)+nt_0]$ and pick a continuously differentiable function $\alpha_n : \mathbb{R}\to\mathbb{R}$ such that

$$\alpha_n(\mu(s_n)) = 1,\ 0\le\alpha_n\le 1,\ supp(\alpha_n)\subset I_n,$$

and

$$\left|\alpha_n'(\mu(s))\right| \le \frac{2}{nt_0},\ \text{for every } s\in\mathbb{R}.$$

Let us now define a function $v_n : \mathbb{R}\to E$ by

$$v_n(s) = \begin{cases} \alpha_n(\mu(s))\, U(s,\mu^{-1}(\mu(s_n)-nt_0))x_n, & s\ge\mu^{-1}(\mu(s_n)-nt_0),\\ 0, & \text{otherwise.}\end{cases}$$

One can easily check that $v_n\in C_c(\mathbb{R},E)$ and $supp(v_n)\subset\mu^{-1}(I_n)$.

Furthermore, from (9) we get

$$\|v_n\|_\infty \ge \|v_n(s_n)\| = \|U(s_n,\mu^{-1}(\mu(s_n)-nt_0))x_n\| > \frac{1}{2}.$$

For $s\ge\mu^{-1}(\mu(s_n)+t-nt_0)$ we have

$$T_t v_n(s) = \alpha_n(\mu(s)-t)\, U(s,\mu^{-1}(\mu(s_n)-nt_0))x_n,$$

thus

$$\frac{1}{t}\left(T_t v_n - v_n\right)(s) = \frac{1}{t}\left(\alpha_n(\mu(s)-t)-\alpha_n(\mu(s))\right) U(s,\mu^{-1}(\mu(s_n)-nt_0))x_n.$$

This yields that $v_n\in D(G)$ and

$$Gv_n(s) = \begin{cases} -\alpha_n'(\mu(s))\, U(s,\mu^{-1}(\mu(s_n)-nt_0))x_n, & s\ge\mu^{-1}(\mu(s_n)-nt_0),\\ 0, & \text{otherwise.}\end{cases}$$

Fix $s\in\mu^{-1}(I_n)$ and put $k = \left[\frac{\mu(s)-\mu(s_n)}{t_0}\right]+n$. It follows that $k\in\{0,1,\dots,2n\}$. Moreover, writing

$$\frac{\mu(s)-\mu(s_n)}{t_0} = \left[\frac{\mu(s)-\mu(s_n)}{t_0}\right] + c_s,\ \text{with } c_s\in[0,1),$$

we get

$$s = \mu^{-1}(\mu(s_n)+(k-n)t_0+c_s t_0).$$

Using (1) and the right-hand side of (8) we have

$$\begin{aligned}
\|Gv_n(s)\| &\le \frac{2}{nt_0}\|U(s,\mu^{-1}(\mu(s_n)-nt_0))x_n\|\\
&= \frac{2}{nt_0}\|U(\mu^{-1}(\mu(s_n)+(k-n)t_0+c_s t_0),\mu^{-1}(\mu(s_n)-nt_0))x_n\|\\
&\le \frac{2}{nt_0}Ke^{\alpha c_s t_0}\|U(\mu^{-1}(\mu(s_n)+(k-n)t_0),\mu^{-1}(\mu(s_n)-nt_0))x_n\|\\
&\le \frac{2}{nt_0}Ke^{\alpha t_0}\|U(\mu^{-1}(\mu(s_n)+(k-n)t_0),\mu^{-1}(\mu(s_n)+(k-n)t_0-kt_0))\,x_n\|\\
&\le \frac{3}{nt_0}Ke^{\alpha t_0},
\end{aligned}$$

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where $x_n = u_{k_n}(\mu^{-1}(\mu(s_n)+(k-n)t_0-kt_0))$. Therefore, $Gv_n \to 0$ in $C_0(\mathbb{R}, E)$, and thus $0 \in A\sigma(G)$. This proves (7) and thus (6) holds. □

Corollary 3.2. *The spectrum $\sigma(T_t)$ is rotationally invariant for $t > 0$, that is*

$$\lambda\sigma(T_t) = \sigma(T_t), \text{ for every } \lambda \in \mathbb{C} \text{ with } |\lambda| = 1. \tag{10}$$

Proof. For each $\gamma \in \mathbb{R}$ we define the isometry

$$\Psi_\gamma u(s) = e^{i\gamma\mu(s)}u(s), \ u \in C_0(\mathbb{R}, E).$$

We have

$$\begin{aligned}\Psi_\gamma T_t \Psi_{-\gamma} u(s) &= e^{i\gamma\mu(s)}U(s, \mu^{-1}(\mu(s)-t))e^{-i\gamma(\mu(s)-t)}u(\mu^{-1}(\mu(s)-t)) \\ &= e^{i\gamma t}T_t u(s).\end{aligned}$$

It is well-known that the generator of the C_0-semigroup $\{\Psi_\gamma T_t \Psi_{-\gamma}\}_{t\geq 0}$ is $\Psi_\gamma G \Psi_{-\gamma}$ and $\sigma(G) = \sigma(\Psi_\gamma G \Psi_{-\gamma})$ (see, for instance, [4, II.2.1]). On the other hand, the generator of $\left\{e^{i\gamma t}T_t\right\}_{t\geq 0}$ is $G + i\gamma \mathrm{I}$ and $\sigma(G + i\gamma \mathrm{I}) = \sigma(G) + i\gamma$ [4, II.2.1]. All these show that

$$\sigma(G) = \sigma(G) + i\gamma, \ \gamma \in \mathbb{R}.$$

Finally, the spectral mapping theorem and the above identity prove (10). □

Comments

Throughout this paper, all the results stand for $C_0(\mathbb{R}, E)$. Evidently they can be extended similarly to the general context of nonuniform behavior, formally replacing the function space $C_0(\mathbb{R}, E)$ with the super-space C_* introduced in [1]. However, such an approach would significantly complicate computations, without adding any essential merit for the main purpose.

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Nicolae Lupa – Department of Mathematics,
Politehnica University of Timişoara,
Piaţa Victoriei 2, 300006 Timişoara, Romania
E-mail: nicolae.lupa@upt.ro

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