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THE DYNAMICS OF AN UNEMPLOYMENT MODEL WITH DISCRETE TIME DELAY AND OPTIMAL CONTROL

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Abstract

The purpose of this paper is to analyzed a mathematical model of three ordinary differential equations which describe the unemployment incorporating a discrete time delay and optimal control. The model's investigation focuses on assessing the stability analysis of the equilibrium point as well as the optimal control. Theoretical statements are supported by conducting numerical simulations. [2]

1 Introduction

Comprehending the dynamic nature of social issues is crucial in developing effective solutions, particularly in the case of unemployment, which is a pervasive concern impacting individuals worldwide. As an essential indicator of a nation's economic status, gaining a comprehensive understanding of this issue is of paramount importance.

In recent years, researchers have explored mathematical models to study unemployment, as documented in various works such as Nikolopoulos and Tzanetis (1999) [15], Misra and Singh (2013) [13], Misra et al. (2017) [14], Pathan and Singh (2017) [16] and Harding (2018) [6]. Initially, Nikolopoulos and Tzanetis (1999) proposed a model, which was further developed by Misra and Singh (2011) to investigate unemployment using three variables. Subsequently, in Misra and Singh (2013), they incorporated time delay and employed the theory of delay differential equations to analyze the model. Building on this concept, Harding and Neamtu (2018) formulated an unemployment model with distributed time delay [6].

Furthermore, prior research in the field has delved into various aspects of three-dimensional unemployment models, encompassing investigations involving discrete time delay [8], distributed time delay [9], and even scenarios with two distinct distributed time delays [18]. In this work is analyzed the model [8] by introducing the optimal control strategy.

These existing mathematical models offer novel approaches to analyze unemployment by leveraging historical data on state variables, which serves as the foundation for the current research work.

[2]MSC(2010): : 34D23, 91B39, 91B55

Keywords and phrases: *unemployment model, stability properties, optimal control.*

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2 The mathematical model

In this study is analyzed the mathematical model of unemployment introduced in [8], incorporating optimal control technique. The paper considers the following aspects:

- Three variables are taken into account: U represents the total number of unemployed individuals, E denotes the total number of employed individuals, and V represents the total number of newly created jobs by the government and private sector.
- The model assumes that all individuals entering the system at time t possess the necessary qualifications for any available job.
- The number of unemployed individuals increases steadily at a constant rate, denoted as A.
- Both the government and private sector exert efforts to generate new job openings, with the number of job opportunities being directly proportional to the number of unemployed individuals.
- It is also assumed that if individuals are fired or voluntarily leave their jobs within the employed group, they transition to the unemployed group.

Based on the previously mentioned considerations, the mathematical model described in [8] is:

$$\begin{cases} \dot{U}(t) = A - [a_1 V(t) + a_2]U(t) + a_3 E(t) - b_1 U(t), \\ \dot{E}(t) = [a_1 V(t) + a_2]U(t) - a_3 E(t) - b_2 E(t), \\ \dot{V}(t) = a_4 U(t-\tau) - b_3 V(t), \end{cases} \tag{1}$$

with the parameters:

- a_1 - the employment rate pertaining to the newly generated job opportunities by both the government and private sector, specifically aimed at the unemployed individuals;
- a_2 - the employment rate for the unemployed individuals in relation to the existing job positions;
- a_3 - the rate at which individuals transition from the employed category to the unemployed category due to being terminated or voluntarily leaving their jobs;
- b_1 - the death rate and the migration rate of unemployed persons;
- b_2 - the rate of death, retirement and migration of employed persons;
- a_4 - the rate at which new job vacancies are being created;
- b_3 - the diminution of the new vacancies created by gouverment and private sectors;
- τ - discrete time delay

The first result for positivity and boundedness of the solution of the system (1) is proved in [8].

Theorem 2.1. *The set,*

$$\Omega = \left\{ (U, E, V) \in \mathbb{R}^3 \ : \ 0 < U + E < \frac{A}{\delta}, \ 0 < V < \frac{a_4 A}{\delta b_3} \right\},$$

where $\delta = min(b_1, b_2)$ is a region of attraction for the system (1) and it attracts all the solutions initiating from the interior of the open positive octant of $\mathbb{R}^3$.

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3 Equilibrium analysis

By solving the system presented as follows:

$$\begin{cases} A-(a_1V+a_2)U+a_3E-b_1U=0 \\ (a_1V+a_2)U-a_3E-b_2E=0 \\ a_4U-b_3V=0 \end{cases} \tag{2}$$

the unique positive equilibrium of the model (1) is determined, which is denoted as S^+. The equilibrium is given by:

$$S^+ := (U_0, E_0, V_0) = \left(U_0, \frac{U_0(a_1a_4U_0+a_2b_3)}{b_3(a_3+b_2)}, \frac{a_4U_0}{b_3}\right).$$

4 Local stability analysis

The characteristic equation for system (1) is:

$$a_1a_4U_0e^{-\lambda\tau}(\lambda+b_2)+\lambda^3+C_2\lambda^2+C_1\lambda+C_0=0, \tag{3}$$

where the coefficients

$$\begin{aligned} C_2 &= a_1V_0+a_2+a_3+b_1+b_2+b_3 \\ C_1 &= a_1b_2V_0+a_1b_3V_0+a_2b_3+a_2b_2+a_3b_1+b_1b_2+b_1b_3+a_3b_3+b_2b_3 \\ C_0 &= a_1b_2b_3V_0+a_2b_2b_3+b_1b_3a_3+b_1b_2b_3 \end{aligned}$$

are positive.

Denoting with $P(\lambda)=\lambda^3+C_2\lambda^2+C_1\lambda+C_0$, the equation (3) becomes:

$$a_1a_4U_0e^{-\lambda\tau}(\lambda+b_2)+P(\lambda)=0 \tag{4}$$

In [8], the following results are proven:

Theorem 4.1. *In the non-delayed case, the equilibrium point of the system (1) S^+ is locally asymptotically stable.*

Lemma 4.2. *Considering $\tau>0$ as bifurcation parameter, system (1) does not undergo a Hopf bifurcation in a neighborhood of the positive equilibrium S^+, for any value of the time delay.*

The previous lemma implies the following important result:

Theorem 4.3. *The positive equilibrium S^+ of system (1) is locally asymptotically stable for any value of the time delay $\tau\geq 0$.*

5 Optimal control strategy

In this section, the necessary and sufficient condition for the optimal control strategy to be presented is outlined.

Firstly, it is considered t_f and Δ as constants. Then, it is defined an admissible control set $U_{ad}=\{u(t)$ which is measurable, $0\leq u(t)\leq\Delta, t\in[0,t_f]\}$. The variable $u(t)$ in this context is referred to as a control variable, representing the strategy employed to decrease the number of

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unemployed individuals. Based on these facts, it is analyzed an optimal control problem aimed at minimizing the objective functional

$$J(u) = \int_0^{t_f} [U(t) + \frac{\epsilon}{2} u^2(t)] dt \tag{5}$$

under the constraint of:

$$\begin{cases} \dot{U}(t) = A - [a_1 V(t) + a_2] U(t) + a_3 E(t) - b_1 U(t) - u(t) U(t), \\ \dot{E}(t) = [a_1 V(t) + a_2] U(t) - a_3 E(t) - b_2 E(t) + \omega(t) u(t) U(t), \\ \dot{V}(t) = a_4 U(t-\tau) - b_3 V(t) - \omega(t) u(t) U(t) + u(t) U(t), \end{cases} \tag{6}$$

To ensure a balanced magnitude of the control variable, denoted as $u(t)$, a small positive constant ϵ is introduced. The introduction of this constant accounts for the squared control variable, which signifies the extent of the size effects of the control.

To obtain an optimal solution, the initial step involves deriving the Lagrangian and Hamiltonian for the optimal control problem stated in equations (5) and (6). Specifically, the Lagrangian of the optimal problem can be represented in the following manner:

$$\tilde{L}(U, u) = U(t) + \frac{\epsilon}{2} u^2(t). \tag{7}$$

In order to identify the optimal control function for the provided optimal control problem, we define the corresponding Hamiltonian as follows:

$$H(U, E, V, u, \lambda_1, \lambda_2, \lambda_3, t) = \tilde{L}(U, u) + \lambda_1(t)\dot{U}(t) + \lambda_2(t)\dot{E}(t) + \lambda_3(t)\dot{V}(t), \tag{8}$$

where λ_1, λ_2, λ_3 are the adjoint functions to be determined suitably. Firstly, we establish the existence of a solution for the control system (6).

As in [3], we have:

Theorem 5.1. *There exists an optimal control $u^*(t)$ such that $J(u^*(t)) = \min J(u(t))$, under the constraint of the control system stated in (6), along with the given initial conditions.*

Proof. It can observed:

1. There is a nonempty set of control and corresponding state variables.
2. The admissible set U_{ad} is both convex and closed.
3. The right-hand side of the state system (6) is bounded by a linear function involving the state variables.
4. The function $\tilde{L}(U, u) = U + \frac{\epsilon}{2} u^2$ is concave when considering the admissible control set U_{ad}.
5. There exists a constant $\rho > 1$, $\eta_1 > 0$ and η_2 such that $\tilde{L}(U, u) \geq \eta_1 (|u|)^\rho + \eta_2$. Specifically, we can choose $\eta_1 = \frac{\epsilon}{2}$ and η_2 represents the lower bound on U, which is similar to the lower bound mentioned in [7]. Thus, it can be directly obtain the result from [2], completing the proof.

□

Moving forward, a necessary condition for the optimal control strategy is derived using Pontryagin's Maximum Principle [4].

Theorem 5.2. *When considering an optimal control variable $u^*(t)$ and the corresponding solution $U^*(t)$, $E^*(t)$, $V^*(t)$ of state system (6), there exists adjoint variables $\lambda_1(t)$, $\lambda_2(t)$ and $\lambda_3(t)$ that satisfy*

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$$\begin{cases} \dot{\lambda}_1(t) = -1 + [a_1V^*(t) + a_2 + b_1 + u^*(t)]\lambda_1(t) - [a_1V^*(t) + a_2 + \omega u^*(t)]\lambda_2(t) - \\ \qquad - (1-\omega)u^*(t)\lambda_3(t) - \chi_{[0,t_f-\tau]}(t)a_4\lambda_3(t+\tau), \\ \dot{\lambda}_2(t) = -a_3\lambda_1(t) + (a_3 + b_2)\lambda_2(t), \\ \dot{\lambda}_3(t) = a_1U^*(t)\lambda_1(t) - a_1U^*(t)\lambda_2(t) + b_3\lambda_3(t), \end{cases} \tag{9}$$

with transversality conditions

$$\lambda_i(t_f) = 0, \quad i = 1, 2, 3.$$

Here $\chi_{[0,t_f-\tau]}(t) = 1$ *if* $t \in [0, t_f - \tau]$. *Otherwise* $\chi_{[0,t_f-\tau]}(t) = 0$. *Furthermore, the optimal control* $u^*(t)$ *is given*

$$u^*(t) = min\{max(0, R), \Delta\}, \tag{10}$$

where

$$R = \frac{U(t+1)}{\epsilon}[\lambda_1(t) - \omega\lambda_2(t) - (1-\omega)\lambda_3(t)].$$

Proof. To derive the adjoint equations and the transversality conditions, we employ the Hamiltonian stated in equation (8). By differentiating the Hamiltonian (8), we acquire the adjoint system in the following manner:

$$\begin{cases} \dot{\lambda}_1(t) = -\frac{\partial H}{\partial U}(t) - \chi_{[0,t_f-\tau]}(t)\frac{\partial H}{\partial U_\tau}(t+\tau), \\ \dot{\lambda}_2(t) = -\frac{\partial H}{\partial E}(t) - \chi_{[0,t_f-\tau]}(t)\frac{\partial H}{\partial E_\tau}(t+\tau), \\ \dot{\lambda}_3(t) = -\frac{\partial H}{\partial V}(t) - \chi_{[0,t_f-\tau]}(t)\frac{\partial H}{\partial V_\tau}(t+\tau). \end{cases} \tag{11}$$

Hence, the adjoint system can be reformulated as system (9). According to the optimal conditions, we obtain the following:

$$\frac{\partial H}{\partial u}(t)|_{u=u^*(t)} = -\epsilon u^*(t) + \omega\lambda_1(t)V^*(t) + (1-\omega)\lambda_2(t)V^*(t) - \lambda_3(t)V^*(t) = 0. \tag{12}$$

Consequently, it can be deduced that $u^*(t) = min\{max(0, R), \Delta\}$. By considering the characteristics of the admissible control set U_{ad}, equation (10) can be obtained and the proof is finalized.

As a result, we have successfully derived the following optimality system:

$$\begin{cases} \dot{\lambda}_1(t) = -1 + [a_1V^*(t) + a_2 + b_1 + u^*(t)]\lambda_1(t) - [a_1V^*(t) + a_2 + \omega u^*(t)]\lambda_2(t) - \\ \qquad - (1-\omega)u^*(t)\lambda_3(t) - \chi_{[0,t_f-\tau]}(t)a_4\lambda_3(t+\tau), \\ \dot{\lambda}_2(t) = -a_3\lambda_1(t) + (a_3 + b_2)\lambda_2(t), \\ \dot{\lambda}_3(t) = a_1U^*(t)\lambda_1(t) - a_1U^*(t)\lambda_2(t) + b_3\lambda_3(t), \\ u^*(t) = min\{max(0, R), \Delta\}, \quad R = \frac{U(t+1)}{\epsilon}[\lambda_1(t) - \omega\lambda_2(t) - (1-\omega)\lambda_3(t)], \\ \lambda_i(t_f) = 0, \quad i = 1, 2, 3. \end{cases} \tag{13}$$

and

$$\begin{cases} \dot{U}^*(t) = A - [a_1V^*(t) + a_2]U^*(t) + a_3E^*(t) - b_1U^*(t) - u^*(t)U^*(t), \\ \dot{E}^*(t) = [a_1V^*(t) + a_2]U^*(t) - a_3E^*(t) - b_2E^*(t) + \omega(t)u^*(t)U^*(t), \\ \dot{V}^*(t) = a_4U^*(t-\tau) - b_3V^*(t) - \omega(t)u^*(t)U^*(t) + u^*(t)U^*(t). \end{cases} \tag{14}$$

□

6 Numerical simulations

The system parameters for the numerical simulations are as follows: A is chosen as 5000, a_1 is set to 0.00002, a_2 is set to 0.4, a_3 is set to 0.01, a_4 is set to 0.007, $b_1 = 0.04$, $b_2 = 0.05$, $b_3 = 0.05$, $\omega = 0.8$, $\Delta = 0.8$, $\epsilon = 20000$ and $t_f = 500$.

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In the initial three instances illustrated in Figure 1, we plotted the evolution of individuals who were unemployed, individuals who were employed, and the number of new jobs generated by both the government and the private sector. These cases included scenarios involving constant high control, optimal control and without control. Based on the findings from Figure 1, it can be concluded that the implemented control measures have proven to be effective.

Figure 2 exhibits the collective evolution of the three variables $U(t)$, $E(t)$, and $V(t)$, alongside the control strategy.

Figure 3 describes the temporal evolution of the state variables $U(t)$, $E(t)$, and $V(t)$ in the absence of the delay and control. Additionally, the figure shows that the equilibrium point of the initial system is asymptotically stable.

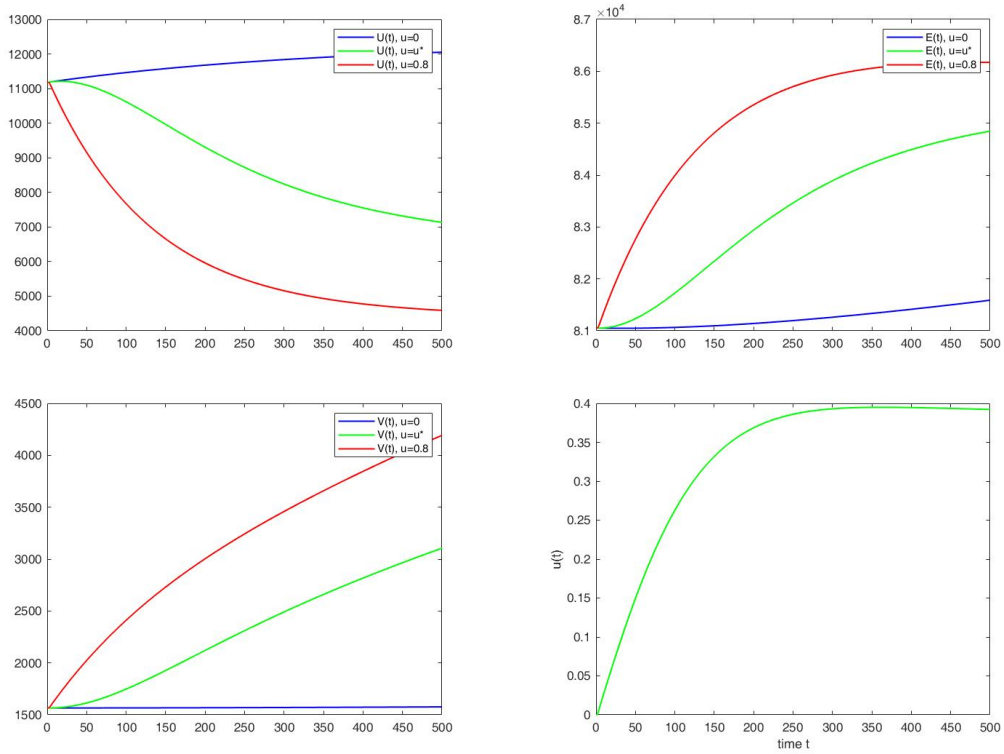


Figure 17: Evolution of state variables $U(t)$, $E(t)$ and $V(t)$ of system (6) with optimal control (red), with high control (red) and without control (blue).

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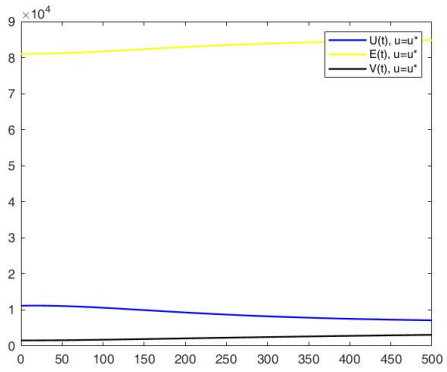


Figure 18: Evolution of state variables $U(t)$ (blue), $E(t)$ (yellow) and $V(t)$ (black) of system (6) with optimal control.

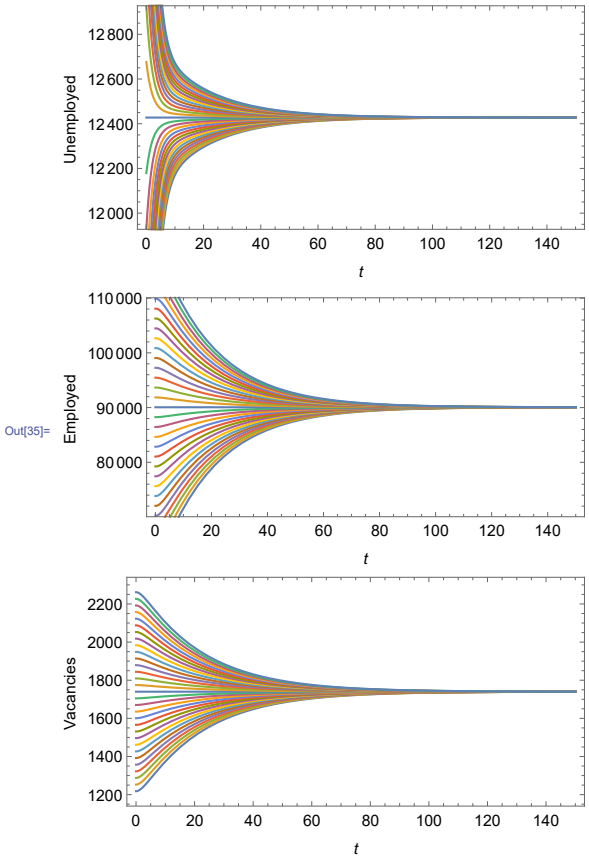


Figure 19: Evolution of state variables $U(t)$, $E(t)$ and $V(t)$ of system (1) without delay and control.

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7 Conclusions

In this paper the model proposed in [8] which incorporates one discrete time delay was investigated. Initially, the local stability results were presented, followed by a discussion on the optimal control.

In particular, this work represented the first instance in which an optimal control problem was formulated for the controlled system (1). Theorems 5.1 and 5.2 present an optimal strategy aimed at minimizing the overall count of unemployed individuals.

Furthermore, upon conducting numerical simulations, it becomes evident that it is reasonable to deduce that the control strategy u^* is, indeed, the optimal selection for minimizing the objective function $J(u)$. This study has the potential to provide a fresh outlook on the issue of unemployment control, among other aspects.

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