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ON THE PERIODIC AND HOMOCLINIC ORBITS OF A HAMILTON-POISSON SYSTEM

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Abstract

In this paper, we study a Hamilton-Poisson system that has a single family of equilibrium points. We show that there are periodic orbits around all the nonlinearly stable equilibrium points, and homoclinic orbits that connect each unstable equilibrium point with itself. We point out these properties in connection with the energy-Casimir mapping associated to the considered system. [1]

1 Introduction

Let Ω be an open set in $\mathbb{R}^3$ and $H, C \in C^\infty(\Omega)$. It is well known that the functions H and C define the three-dimensional system of differential equations

$$\dot{\mathbf{x}} = \nabla H \times \nabla C, \tag{1}$$

for which they are constants of motion. Moreover, (1) is a Hamilton-Poisson system, where the Poisson bracket in the smooth category is given by (see, e.g., [14])

$$\{f, g\} = \nabla C \cdot (\nabla f \times \nabla g). \tag{2}$$

This Poisson bracket can be written in matrix notation

$$\Pi = \begin{bmatrix} 0 & C_z' & -C_y' \\ -C_z' & 0 & C_x' \\ C_y' & -C_x' & 0 \end{bmatrix}. \tag{3}$$

[1] MSC(2010): 70H12, 70H14, 70K20, 70K42, 70K44.
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The study of the common level sets of the independent constants of motion H and C may gives informations about the dynamics of (1). For instance, Holm and Marsden described such results in the case of rigid body dynamics [6].

In [15], the energy-Casimir mapping associated to a three-dimensional Hamilton-Poisson system was considered. If H is the Hamiltonian of the system and C is a Casimir function of the Poisson structure, then the energy-Casimir mapping is defined by

$$\mathcal{EC} : \mathbb{R}^3 \to \mathbb{R}^2 \ , \quad \mathcal{EC}(x, y, z) = (H(x, y, z), C(x, y, z)). \tag{4}$$

In order to classify the types of the orbits of system (1), a partition of the image of the energy-Casimir mapping given by the images of the equilibrium points through the energy-Casimir mapping was constructed [15] (also see [4, 5, 7, 8, 9, 10, 11, 12, 13, 16, 17]). In these papers, the considered Hamilton-Poisson systems have at least two families of equilibrium points, which lead to a partition of the image of $\mathcal{EC}$. Moreover, a topological classification of the fibers and the corresponding dynamical description is obtained. The aim of our work is to verify whether the connections between the above-mentioned partition and the dynamics of the system, reported in the above papers, hold for a system with a single family of equilibrium points. More precisely, we obtain that the boundary of the set $\mathrm{Im}(\mathcal{EC}) \subsetneq \mathbb{R}^2$ is the union of the images of some stable equilibrium points through $\mathcal{EC}$, and the image of the energy-Casimir mapping is convexly generated by these images. Furthermore, the fibers $\mathcal{F}_{(h,c)} = \mathcal{EC}^{-1}(h, c)$ corresponding to the pairs (h, c) that belong to the boundary of $\mathrm{Im}(\mathcal{EC})$ contain only nonlinearly stable equilibrium points. The images of the others critical points are in the interior of $\mathrm{Im}(\mathcal{EC})$ and the corresponding fibers are periodic orbits and homoclinic orbits for stable and unstable equilibria respectively. In addition, if Σ denotes an open subset of the semialgebraic partition of $\mathrm{Im}(\mathcal{EC})$ that has dimension 2 and $(h, c) \in \Sigma$, then the fiber $\mathcal{F}_{(h,c)}$ contains periodic orbits.

In this paper we study the system defined by the functions

$$H(x, y, z) = \frac{1}{4}x^2 + \frac{1}{4}y^2 - z \ , \quad C(x, y, z) = \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z^2, \tag{5}$$

that is

$$\begin{cases} \dot{x} = \dfrac{1}{2}(1 + yz) \\ \dot{y} = -\dfrac{1}{2}(1 + xz) \\ \dot{z} = \dfrac{1}{4}(x - y) \end{cases} \tag{6}$$

The paper is set up as follows. In Section 2, we analyze the nonlinear stability of the equilibrium points of the considered system. In Section 3, we prove the existence of periodic orbits around all the nonlinearly stable equilibrium points.

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In Section 4, we give two Hamilton-Poisson realizations of the considered system, which lead to an infinite family of Hamilton-Poisson realizations. In Section 5, we study some properties of the energy-Casimir mapping in connection with the dynamics of system (6). Particularly, we deduce parametric equations for some periodic and homoclinic orbits.

2 Stability

In this section we study the stability of the equilibrium points of system (6).

The equilibrium points of the system (6) satisfy the conditions

$$\begin{cases} \frac{1}{2}(1+yz)=0 \\ -\frac{1}{2}(1+xz)=0 \\ \frac{1}{4}(x-y)=0 \end{cases}.$$

Solving this system, we get the following family of equilibrium points of system (6)

$$\mathcal{E}=\left\{\left(M,M,-\frac{1}{M}\right)|M\in\mathbb{R}^*\right\}. \tag{7}$$

We obtain the following result regarding the stability of these equilibrium points.

Proposition 2.1. *Let $E_M=\left(M,M,-\frac{1}{M}\right)\in\mathcal{E}$ an arbitrary equilibrium point. Then E_M is unstable for every $M\in[1,+\infty)$ and nonlinearly stable for every $M\in(-\infty,0)\cup(0,1)$.*

Proof. Let $M\in\mathbb{R}^*$, the equilibrium point $E_M=\left(M,M,-\frac{1}{M}\right)$ and $J(x,y,z)$ be the matrix of linear part of system (6), that is

$$J(x,y,z)=\begin{bmatrix} 0 & \frac{1}{2}z & \frac{1}{2}y \\ -\frac{1}{2}z & 0 & -\frac{1}{2}x \\ \frac{1}{4} & -\frac{1}{4} & 0 \end{bmatrix} \tag{8}$$

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The characteristic polynomial of $J(E_M)$ is given by

$$P_{J(E_M)}(\lambda) = -\lambda\left(\lambda^2 - \frac{M^3-1}{4M^2}\right) \tag{9}$$

and the roots of $P_{J(E_M)}$ are

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm\frac{\sqrt{M^3-1}}{2M} \tag{10}$$

For $M \in (1, +\infty)$ there is a positive eigenvalue and the equilibrium point E_M is unstable.

In the case $M = 1$, the equilibrium point E_M becomes $E_1(1, 1, -1)$ and the roots of the characteristic polynomial are all zero. We will show that there is a homoclinic orbit that connects E_1 with itself.

We remind that the functions H and C (5) are constants of motion, so we can define implicit equation of an orbit of system (6) by considering the level sets $H(x, y, z) = constant$ and $C(x, y, z) = constant$. Particularly, for the equilibrium point $E_1(1, 1, -1)$ we have

$$\begin{cases} H(x,y,z) = H(1,1,-1) \\ C(x,y,z) = C(1,1,-1) \end{cases},$$

which is equivalent to

$$\begin{cases} x^2 + y^2 - 4z = 6 \\ x + y + z^2 = 3 \end{cases}.$$

We get

$$x - y = \pm\sqrt{3 + 8z + 6z^2 - z^4}.$$

To obtain solution of system (6), we can reduce it from three degrees of freedom to one degree of freedom by rewriting the equation $\dot{z} = \frac{1}{4}(x - y)$ as

$$\dot{z} = \pm\frac{1}{4}\sqrt{3 + 8z + 6z^2 - z^4}. \tag{11}$$

Therefore

$$\int \frac{dz}{\sqrt{3 + 8z + 6z^2 - z^4}} = \pm\frac{1}{4}t.$$

Denoting $q = z + 1$, the above integral writes

$$\int \frac{dz}{\sqrt{3 + 8z + 6z^2 - z^4}} = \int \frac{dq}{q^2\sqrt{\frac{4}{q} - 1}}$$

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Hence by integrating the equation (11), we get

$$-\frac{1}{2}\sqrt{\frac{4}{1+z}-1}=\pm\frac{1}{4}t$$

so we obtain

$$z(t)=\frac{12-t^2}{4+t^2} \tag{12}$$

Therefore we deduce

$$x(t)=\frac{t^4+24t^2-64t-48}{(4+t^2)^2}\ ,\ y(t)=\frac{t^4+24t^2+64t-48}{(4+t^2)^2}. \tag{13}$$

As a result, we get a solution of system (6).

Since $\lim\limits_{t\to\pm\infty}(x(t),y(t),z(t))=(1,1,-1)$, we get the homoclinic orbit

$$\mathcal{H}=(x,y,z):\mathbb{R}\to\mathbb{R}^3,\mathcal{H}(t)=(x(t),y(t),z(t)), \tag{14}$$

where the functions x,y,z are given by (12), (13).

For this reason, the equilibrium point $(1,1,-1)$ is unstable.

For $M\in(0,1)$, we will study stability of the equilibrium point E_M using the Arnold test [1] (also see [2]). We consider the function

$$F_\lambda(x,y,z)=C(x,y,z)-\lambda H(x,y,z),$$

where λ is a real parameter. Then we have successively the following:

1. $dF_\lambda\left(M,M,-\frac{1}{M}\right)=0$ if and only if $\lambda=\frac{1}{M}$.
2. $W=ker\ dC\left(M,M,-\frac{1}{M}\right)=span_{\mathbb{R}}\left\{\left(1,0,\frac{M}{2}\right),\left(0,1,\frac{M}{2}\right)\right\}$.
3. $d^2F_\lambda\big|_{W\times W}=\left(\frac{M^3-2}{4M}\right)dx^2+\frac{M^2}{2}dxdy+\left(\frac{M^3-2}{4M}\right)dy^2$ is negative definite for $M\in(0,1)$.

Hence, from the Arnold stability test we conclude that E_M is nonlinearly stable for $M\in(0,1)$.

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To study the stability of the equilibrium point E_M in the case $M \in (-\infty, 0)$, we will consider the Lyapunov function $L \in C^\infty(\mathbb{R}^3, \mathbb{R})$,

$$L(x,y,z) = -\frac{1}{4M}(x-M)^2 - \frac{1}{4M}(y-M)^2 + \frac{1}{2}\left(z+\frac{1}{M}\right)^2.$$

Obviously, $L(x,y,z) > 0$, $\forall (x,y,z) \neq E_M$ and $L(E_M) = 0$, so L is positive definite. Moreover, $\dot{L} = \nabla L \cdot \dot{\mathbf{x}} = 0$, where $\dot{\mathbf{x}} = (\dot{x}, \dot{y}, \dot{z})$ is given by (6). Therefore the equilibrium point E_M is nonlinearly stable, which ends the proof. □

3 Periodic orbits

In this section we establish the existence of periodic orbits around the nonlinearly stable equilibrium points of system (6). We shall use the following version of the Moser-Weinstein theorem in the case of zero eigenvalue [3]:

Theorem 3.1. *([3]) Let $\dot{x} = X(x)$ be a dynamical system on a differentiable manifold U, x_0 an equilibrium point, i.e., $X(x_0) = 0$ and $C := (C_1, ..., C_j): U \to \mathbb{R}^j$ a vector valued constant of motion for the above dynamical system with $C(x_0)$ a regular value for C. If*
(i) the eigenspace corresponding to the eigenvalue zero of the linearized system around x_0 has dimension j,
(ii) $DX(x_0)$ has a pair of pure complex eigenvalues $\pm i\omega$ with $\omega \neq 0$,
(iii) there exist a constant of motion $I: U \to \mathbb{R}$ for the vector field X with $dI(x_0) = 0$ and such that $d^2I(x_0)|_{W\times W} > 0$, where $W = \bigcap_{l=1}^{j} \ker dC_l(x_0)$,
then for each sufficiently small $\varepsilon > 0$ any integral surface $I(x) = I(x_0) + \varepsilon^2$ contains at least one periodic solution of X whose period is close to the period of the corresponding linear system around x_0.

Using this theorem, we obtain the next result.

Proposition 3.2. *Let $E_M = \left(M, M, -\frac{1}{M}\right) \in \mathcal{E}$ be a nonlinearly stable equilibrium point of system (6) such that $M \in (-\infty, 0)$. Then for each sufficiently small $\varepsilon \in \mathbb{R}_+^*$, any integral surface*

$$\Sigma_\varepsilon^{E_M}: \; -\frac{1}{4M}(x-M)^2 - \frac{1}{4M}(y-M)^2 + \frac{1}{2}\left(z+\frac{1}{M}\right)^2 = \epsilon^2$$

contains at least one periodic orbit $\gamma_\varepsilon^{E_M}$ of system (6) whose period is close to $\frac{2\pi}{\omega}$, where $\omega = \frac{\sqrt{1-M^3}}{2|M|}$.

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Proof. The characteristic polynomial associated with the linearization of system (6) at E_M has the eigenvalues $\lambda_1 = 0$ and $\lambda_{2,3} = \pm i\frac{\sqrt{1-M^3}}{2M}$. The eigenspace corresponding to the eigenvalue zero is $\operatorname{span}_{\mathbb{R}}\left\{(1, 1, \frac{1}{M^2})\right\}$, thus it has dimension 1.

Consider the constant of motion of system (6) given by

$$I(x, y, z) = \frac{1}{2}\left(x + y + z^2\right) - \frac{1}{M}\left(\frac{1}{4}x^2 + \frac{1}{4}y^2 - z\right).$$

We have

1. $dI(M, M, -\frac{1}{M}) = 0$.
2. $d^2I\left(M, M, -\frac{1}{M}\right)\big|_{W\times W} = \left(\frac{M^3-2}{4M}\right)dx^2 + \frac{M^2}{2}dxdy + \left(\frac{M^3-2}{4M}\right)dy^2 > 0,$
   where $W = \ker dC\left(M, M, -\frac{1}{M}\right) = \operatorname{span}_{\mathbb{R}}\left\{\left(1, 0, \frac{M}{2}\right), \left(0, 1, \frac{M}{2}\right)\right\}$.

Using Theorem 3.1, the conclusion follows. □

Similarly, we get the existence of periodic orbits around the other nonlinearly stable equilibrium points.

Proposition 3.3. *Let $E_M = \left(M, M, -\frac{1}{M}\right) \in \mathcal{E}$ be a nonlinear stable equilibrium point of system (6) such that $M \in (0, 1)$. Then for each sufficiently small $\varepsilon \in \mathbb{R}_+^*$, any integral surface*

$$\Sigma_\varepsilon^{E_M} \ : \ \frac{1}{4M}(x - M)^2 + \frac{1}{4M}(y - M)^2 - \frac{1}{2}\left(z + \frac{1}{M}\right)^2 = \epsilon^2$$

contains at least one periodic orbit $\gamma_\varepsilon^{E_M}$ of system (6) whose period is close to $\frac{2\pi}{\omega}$, where $\omega = \frac{\sqrt{1-M^3}}{2M}$.

4 Hamilton-Poisson realizations

In this section we prove that the considered system has infinitely many Hamilton-Poisson realizations.

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Proposition 4.1. *The system* (6) *has the Hamilton-Poisson realization* $(\mathbb{R}^3, \Pi_1, H)$, *where*

$$\Pi_1 = \begin{bmatrix} 0 & z & -\frac{1}{2} \\ -z & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \tag{15}$$

is the matrix representation of the Poisson structure and H *is the Hamilton function defined in* (5).

Proof. Considering the Casimir function C given by (5), the matrix Π (3) becomes Π_1 and $\Pi_1 \cdot \nabla C = \mathbf{0}$. It is easy to see that system (6) writes $\Pi_1 \cdot \nabla H = \dot{\mathbf{x}}^t$, where $\mathbf{x} = (x, y, z)$, as required. □

Proposition 4.2. *The system* (6) *has another Hamilton-Poisson realization* $(\mathbb{R}^3, \Pi_2, C)$, *where the Poisson structure is given by*

$$\Pi_2 = \begin{bmatrix} 0 & 1 & \frac{1}{2}y \\ -1 & 0 & -\frac{1}{2}x \\ -\frac{1}{2}y & \frac{1}{2}x & 0 \end{bmatrix} \tag{16}$$

and C *is defined in* (5). *Moreover, system* (6) *is a bi-Hamiltonian system.*

Proof. Hence Π_2 verify $\Pi_2 \cdot \nabla C = \dot{\mathbf{x}}^t$ and $\Pi_2 \cdot \nabla H = \mathbf{0}$. We can observe that $\Pi_1 \cdot \nabla H = \Pi_2 \cdot \nabla C = \dot{\mathbf{x}}^t$. In addition, Π_1 and Π_2 are compatible Poisson structures, i.e. their sum is also a Poisson structure, thus (6) is a bi-Hamiltonian system. □

Proposition 4.3. *The system* (6) *has a family of Hamilton-Poisson realizations, namely* $(\mathbb{R}^3, \Pi_{a,b}, H_{c,d})$, *where the Hamiltonian function is given by*

$$H_{c,d} = \frac{d}{4}(x^2 + y^2) + \frac{c}{2}(x + y) + \frac{c}{2}z^2 - dz,$$

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and the Poisson structure by

$$\Pi_{a,b} = \begin{bmatrix} 0 & az-b & -\frac{1}{2}a-\frac{1}{2}by \\ -az+b & 0 & \frac{1}{2}a+\frac{1}{2}bx \\ \frac{1}{2}a+\frac{1}{2}by & -\frac{1}{2}a-\frac{1}{2}bx & 0 \end{bmatrix},$$

for every $a,b,c,d \in \mathbb{R}$ that fulfill the condition $ad-bc=1$. A Casimir of the Poisson structure is

$$C_{a,b} = \frac{b}{4}(x^2+y^2) + \frac{a}{2}(x+y) + \frac{a}{2}z^2 - bz.$$

Proof. Let $a,b,c,d \in \mathbb{R}$ such that $ad-bc=1$. Let us denote $\Pi_{a,b} = a\Pi_1 - b\Pi_2$, $H_{c,d} = cC+dH$, and $C_{a,b} = aC+bH$, where Π_1 and Π_2 are given by (15) and (16) respectively. We observe that $\Pi_{a,b}$ verifies $\Pi_{a,b}\cdot\nabla H_{c,d} = \dot{\mathbf{x}}^t$ and $\Pi_{a,b}\cdot\nabla C_{a,b} = \mathbf{0}$ and the conclusion follows. □

5 Energy-Casimir mapping

In this section, we point out some connections between some properties of the energy-Casimir mapping (4) and the dynamics of the considered system (6).

In our case, the Hamiltonian H and a Casimir function C are given by (5). Then the energy-Casimir mapping is given by

$$\mathcal{EC} : \mathbb{R}^3 \to \mathbb{R}^2 \ , \quad \mathcal{EC}(x,y,z) = \left(\frac{1}{4}x^2+\frac{1}{4}y^2-z,\ \frac{1}{2}x+\frac{1}{2}y+\frac{1}{2}z^2\right). \tag{17}$$

Recall that a point $(x_0,y_0,z_0) \in \mathbb{R}^3$ is a critical point of the energy-Casimir mapping if the rank of the Jacobian matrix of $\mathcal{EC}$ at this point is less than 2.

Remark 5.1. *Since system* (6) *has the form* (1)*, it results that the critical points of $\mathcal{EC}$ are the equilibrium points* (7) *of the considered system.*

The image of the equilibrium point $E_M\left(M,M,-\frac{1}{M}\right) \in \mathcal{E}$ trough the energy-Casimir mapping, given by

$$\mathcal{EC}(E_M) = \left(\frac{1}{2}M^2+\frac{1}{M}\ ,\ M+\frac{1}{2M^2}\right), \tag{18}$$

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for every $M \in \mathbb{R}^*$, is a critical value of $\mathcal{EC}$. For $M < 0$, we consider the following curve of critical values

$$\Gamma : \begin{cases} h = \dfrac{1}{2}M^2 + \dfrac{1}{M} \\ c = M + \dfrac{1}{2M^2} \end{cases}, M < 0. \tag{19}$$

It immediately follows that Γ is a simple regular curve that defines a convex bijective function $c : \mathbb{R} \to \mathbb{R}, c = c(h)$ (Figure 1). For each $M < 0$, we denote a point of Γ by (h_e, c_e), where $h_e = \dfrac{1}{2}M^2 + \dfrac{1}{M}$, $c_e = M + \dfrac{1}{2M^2}$.

The image of the energy-Casimir mapping is the set

$$\mathrm{Im}(\mathcal{EC}) = \left\{(h, c) \in \mathbb{R}^2 | (\exists)(x, y, z) \in \mathbb{R}^3 : \mathcal{EC}(x, y, z) = (h, c)\right\}.$$

Now we can prove the next result.

Proposition 5.2. *Let $M < 0$ and $\mathcal{EC}$ (17) be the energy-Casimir mapping of system (6). Then the boundary of Im$(\mathcal{EC})$ is Γ and Im$(\mathcal{EC}) = \overline{co}\{\Gamma\}$, where $\overline{co}$ denoted the convex hull (Figure 1).*

Proof. A pair (h, c) belongs to the image of the energy-Casimir mapping if and only if the system

$$\begin{cases} h = \dfrac{1}{4}x^2 + \dfrac{1}{4}y^2 - z \\ c = \dfrac{1}{2}x + \dfrac{1}{2}y + \dfrac{1}{2}z^2 \end{cases} \tag{20}$$

or equivalent

$$\begin{cases} x^2 + y^2 - 4z = 4h \\ x + y + z^2 = 2c \end{cases} \tag{21}$$

has at least a solution.

First, it is obvious that the above system has solution for $h = h_e$ and $c = c_e$.

Now we show that there are no pairs $(h, c) \in \mathrm{Im}(\mathcal{EC})$ for $h = h_e$ and $c < c_e$, that is system (20) has no solution in this case.

Suppose that system (20) has solution for $h = h_e$ and $c < c_e$. Using (21) we get

$$(x - M)^2 + (y - M)^2 - 2M\left(z + \frac{1}{M}\right)^2 = 4M^2 + \frac{2}{M} - 4Mc,$$

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or equivalent

$$(x-M)^2+(y-M)^2-2M\left(z+\frac{1}{M}\right)^2=4M(c_e-c). \tag{22}$$

But $M<0$ and $c<c_e$, which leads to a contradiction with (22). So for $h=h_e$ and $c<c_e$ system (20) has no solution.

Finally, let $h=h_e$ and $c\geq c_e$. Using (22), system (21) is equivalent with

$$\begin{cases} x^2+y^2-4z=2M^2+\dfrac{4}{M} \\ (x-M)^2+(y-M)^2-2M\left(z+\dfrac{1}{M}\right)^2=4M(c_e-c) \end{cases},$$

which has solution because it that represents the intersection between an ellipsoid with the center $E_M\left(M,M,-\frac{1}{M}\right)$ and a circular paraboloid that contains the point E_M. Therefore system (20) has solution for $h=h_e$ and $c\geq c_e$, that is the points situated above the curve Γ belong to $\mathrm{Im}(\mathcal{EC})$, which finishes the proof. □

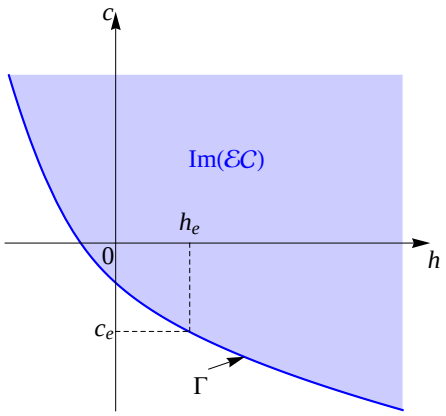


Figure 1: The image of the energy-Casimir mapping.

The images (18) of the equilibrium points through the energy-Casimir mapping allows us to define other two curves of critical values

$$\Gamma^s:\begin{cases} h=\dfrac{1}{2}M^2+\dfrac{1}{M} \\ c=M+\dfrac{1}{2M^2} \end{cases},M\in(0,1)\,,\quad \Gamma^u:\begin{cases} h=\dfrac{1}{2}M^2+\dfrac{1}{M} \\ c=M+\dfrac{1}{2M^2} \end{cases},M\in[1,\infty), \tag{23}$$

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where "s" and "u" stand for stable and unstable respectively. These curves lead to the following semialgebraic partition of the set $\mathrm{Im}(\mathcal{EC})$ depicted in Figure 2

$$\mathrm{Im}(\mathcal{EC}) = \Gamma \cup \Sigma_1 \cup \Gamma^s \cup \Sigma_2 \cup \Gamma^u.$$

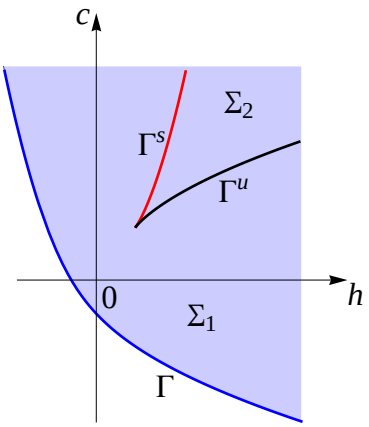


Figure 2: The semialgebraic partition of the image of the energy-Casimir mapping.

In the following, we point out connections between different types of the orbits of system (6) and pairs (h, c) from the subsets of the above partition of $\mathrm{Im}(\mathcal{EC})$.

The fiber of the energy-Casimir mapping $\mathcal{EC}$ corresponding to an element $(h_0, c_0) \in \mathrm{Im}(\mathcal{EC})$ is the set

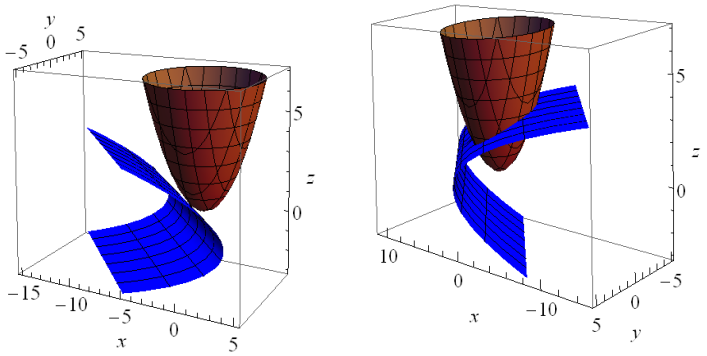
$$\mathcal{F}_{(h_0,c_0)} = \left\{(x, y, z) \in \mathbb{R}^3 \mid \mathcal{EC}(x, y, z) = (h_0, c_0)\right\} \tag{24}$$

and an implicit equation of it is given by

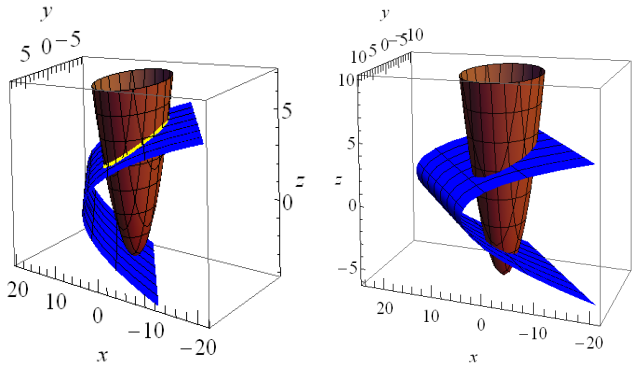
$$\mathcal{F}_{(h_0,c_0)} : \begin{cases} H(x, y, z) = h_0 \\ C(x, y, z) = c_0 \ . \end{cases} \tag{25}$$

Since the dynamics of our system takes place at the intersection of the level sets $H(x, y, z) = constant$ and $C(x, y, z) = constant$, an orbit is given implicitly by (25). In Figure 3 we show such intersections.

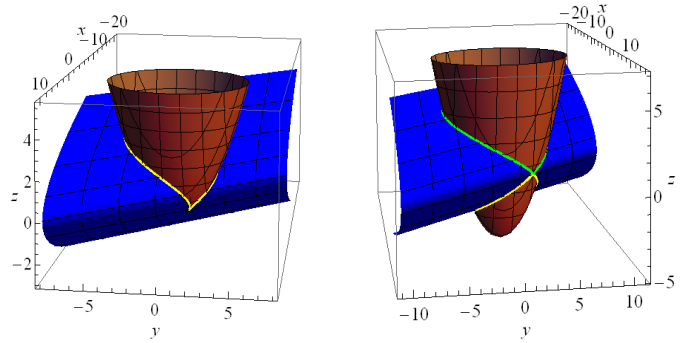
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(a) $(h, c) \in \Gamma$ (b) $(h, c) \in \Sigma_1$



(c) $(h, c) \in \Gamma^s$ (d) $(h, c) \in \Sigma_2$



(e) $(h, c) \in \Gamma^u, M = 1$ (f) $(h, c) \in \Gamma^u, M > 1$

Figure 3: Intersections of the level sets $H(x, y, z) = h$ and $C(x, y, z) = c$:
(a) a stable equilibrium point; (b) a periodic orbit; (c) a stable equilibrium point and a periodic orbit; (d) two periodic orbits; (e) a homoclinic orbit; (f) a pair of homoclinic orbits.

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We begin our study with the pairs (h,c) that belong to the boundary of $\mathrm{Im}(\mathcal{EC})$.

Proposition 5.3. *Let $(h,c)\in\Gamma$. Then the corresponding fiber contains only nonlinear stable equilibrium point,*

$$\mathcal{F}_{(h,c)}=\left\{\left(M,M,-\frac{1}{M}\right)\Big|M\in(-\infty,0)\right\}.$$

Proof. In this case, (5) and (25) lead to

$$-\frac{1}{4M}(x-M)^2-\frac{1}{4M}(y-M)^2+\frac{1}{2}\left(z+\frac{1}{M}\right)^2=c-\frac{1}{M}h-\frac{M}{2}+\frac{1}{2M^2}$$

Since $h=\frac{1}{2}M^2+\frac{1}{M}$, $c=M+\frac{1}{2M^2}$ (19), we get $c-\frac{1}{M}h-\frac{M}{2}+\frac{1}{2M^2}=0$ and

$$-\frac{1}{4M}(x-M)^2-\frac{1}{4M}(y-M)^2+\frac{1}{2}\left(z+\frac{1}{M}\right)^2=0$$

Therefore $x=y=M$ and $z=-\frac{1}{M}$, as required. □

In the proof of Proposition 2.1 we have determined the parametric equations of a homoclinic orbit (see Figure 3(e)) in connection with the equilibrium point $E_1(1,1,-1)$ and the critical value $(h,c)=\mathcal{EC}(E_1)\in\Gamma^u$. More precisely, we have obtained the following result.

Proposition 5.4. *Let $(h,c)=\mathcal{EC}(E_1)\in\Gamma^u$. Then the corresponding fiber contains the unstable equilibrium point $E_1(1,1,-1)$ and the homoclinic orbit $\mathcal{H}$ (14).*

Figure 3(f) indicates a pair of homoclinic orbits corresponding to the unstable equilibrium point E_M, $M>1$. Their parametrizations are given in the next result.

Proposition 5.5. *Let $(h,c)=\mathcal{EC}(E_M)\in\Gamma^u, M>1$. Then the corresponding fiber contains the unstable equilibrium point E_M and a pair of homoclinic orbits $\mathcal{H}_1$ and $\mathcal{H}_2$ given by*

$$\mathcal{H}_1(t)=(x_1(t),y_1(t),z_1(t))\ ,\ \mathcal{H}_2(t)=(x_2(t),y_2(t),z_2(t))\ ,\ t\in\mathbb{R},$$

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where

$$x_1(t)=\frac{M^3+4p-4\left(-1+M^3\right)^{3/2}p+12M^3p^2-6M^6p^2+4M^3p^3+4M^3\left(-1+M^3\right)^{3/2}p^3+M^9p^4}{\left(M+2Mp+M^4p^2\right)^2}$$

$$y_1(t)=\frac{M^3+4p+4\left(-1+M^3\right)^{3/2}p+12M^3p^2-6M^6p^2+4M^3p^3-4M^3\left(-1+M^3\right)^{3/2}p^3+M^9p^4}{\left(M+2Mp+M^4p^2\right)^2}$$

$$z_1(t)=\frac{-1+2p-4M^3p-M^3p^2}{M(1+2p+M^3p^2)}$$

and

$$x_2(t)=\frac{M^3-4p+4\left(-1+M^3\right)^{3/2}p+12M^3p^2-6M^6p^2-4M^3p^3-4M^3\left(-1+M^3\right)^{3/2}p^3+M^9p^4}{\left(M-2Mp+M^4p^2\right)^2}$$

$$y_2(t)=\frac{M^3-4p-4\left(-1+M^3\right)^{3/2}p+12M^3p^2-6M^6p^2-4M^3p^3+4M^3\left(-1+M^3\right)^{3/2}p^3+M^9p^4}{\left(M-2Mp+M^4p^2\right)^2}$$

$$z_2(t)=\frac{-1-2p+4M^3p-M^3p^2}{M(1-2p+M^3p^2)},$$

with $p=p(t)=e^{\frac{\sqrt{-1+M^3}}{2M}(t-t_0)}$.

Proof. The implicit equation of the fiber (25) leads to

$$x-y=\pm\sqrt{8h+8z-4c^2+4cz^2-z^4}.$$

Let $M>1$ and $(h,c)\in\Gamma^u$ (23). Then

$$x-y=\pm\frac{1}{M^2}(1+Mz)\sqrt{4M^3-(Mz-1)^2},$$

and the last equation of system (6) becomes

$$\dot{z}=\pm\frac{1}{4M^2}(1+Mz)\sqrt{4M^3-(Mz-1)^2}.$$

Then

$$\int\frac{4M^2\,dz}{(1+Mz)\sqrt{4M^3-(Mz-1)^2}}=\pm(t-t_0).$$

By changing of variable $z=\dfrac{1}{v}-\dfrac{1}{M}$ we have

$$\int\frac{4dv}{\sqrt{4Mv^2-(1-\frac{2v}{M})^2}}=\mp(t-t_0). \tag{26}$$

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Taking into account that $M > 1$, by integration we get the solutions $z_1(t)$ and $z_2(t)$. Using again (25) in this case, we obtain $x_1(t), y_1(t), x_2(t), y_2(t)$. It is easy to see that $\lim\limits_{t\to\pm\infty}(x_1(t), y_1(t), z_1(t)) = \lim\limits_{t\to\pm\infty}(x_2(t), y_2(t), z_2(t)) = E_M$, which finishes the proof. □

The next result regards the periodic orbit depicted in Figure 3(c).

Proposition 5.6. *Let $(h, c) = \mathcal{EC}(E_M) \in \Gamma^s$. Then the corresponding fiber contains the nonlinearly stable equilibrium point E_M and a periodic orbit $\mathcal{O}$ given by*

$$\mathcal{O}(t) = (x(t), y(t), z(t)) \ , \ t \in \mathbb{R},$$

where

$$x(t) = \frac{M^{3/2}\left(3-2M^3\right)-2\left(1-M^3\right)^{3/2}\cos q+\sin q\left(2+M^{9/2}\sin q\right)}{\sqrt{M}\left(1+M^{3/2}\sin q\right)^2}$$

$$y(t) = \frac{M^{3/2}\left(3-2M^3\right)+2\left(1-M^3\right)^{3/2}\cos q+\sin q\left(2+M^{9/2}\sin q\right)}{\sqrt{M}\left(1+M^{3/2}\sin q\right)^2}$$

$$z(t) = -\frac{1}{M} + \frac{2\left(1-M^3\right)}{M\left(1+M^{3/2}\sin q\right)},$$

with $q = q(t) = \frac{\sqrt{1-M^3}}{2M}(t - t_0)$.

Proof. As in the proof of Proposition 5.5, we obtain equation (26). Taking into account that $0 < M < 1$, we integrate this equation and get the solution $z(t)$, and then $x(t), y(t)$. The presence of the functions $\sin q(t)$ and $\cos q(t)$ shows the periodicity of the orbit. □

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