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PIECEWISE POLYNOMIAL LEAST SQUARES METHOD FOR NONLINEAR HEAT TRANSFER PROBLEMS

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Abstract

In this paper is used a recently introduced approximation method, namely the Piecewise Polynomial Least Squares Method (PWPLSM), in order to compute analytical approximate polynomial solutions for several nonlinear heat transfer problems.

Analyzing the errors obtained by applying the Piecewise Polynomial Least Squares Method with those found in the literature, the accuracy of the method is illustrated. [1]

1 Introduction

The heat transfer phenomena are mainly nonlinear and thus they are best modeled by using nonlinear equations. These equations most of the time can not be solved analytically using traditional methods and, when the numerical solutions are not sufficient, an approximate analytical solution must be computed. In recent years many methods have been developed to compute approximate solutions for nonlinear heat transfer problems. Is mentioned only some of the methods that led to obtaining approximate solutions for this type of equations:

- the Homotopy perturbation method (HPM)([2],[1],[3]),

[1]Mathematical Subject Classification(2010):*34K28, 45L05*
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- the Optimal Homotopy Asymptotic Method (OHAM)([4]),
- the Homotopy analysis method (HAM)([5],[6]),
- the Squared remainder minimization method (SRMM)([7]),
- the Generalized approximation method (GAM)([8]),
- the Least Squares Differential Quadrature Method (LSDQM) ([9]).

In this paper is computed approximate analytical solutions for some well-known nonlinear heat transfer problems ([16],[17],[18]) modeled by using nonlinear ordinary differential equations.

In the second section is described the Piecewise Polynomial Least Squares Method (PWPLSM), which is used for determine analytical approximate polynomial solutions for the above-mentioned type of problems. In the last section with numerical application, is compared approximate solutions obtained by using PWPLSM with previous approximate solutions presented in ([4],[8],[7],[10],[9]). The computations show that by using Piecewise Polynomial Least Squares Method we obtain approximations with an error smaller than the errors obtained by using other methods thus the usefulness of the method being demonstrated.

2 Piecewise Polynomial Least Squares Method (PWPLSM)

We consider a problem consisting of a nonlinear differential equation of order n:

$$y^{(n)}(x) = F(y^{(n-1)}(x), y^{(n-2)}(x), \cdots, y^{(1)}(x), y(x), x) \tag{1}$$

where F is a continuous function, y is an absolutely continuous function, $x \in [a, b]$ and the boundary conditions:

$$\begin{aligned} d_{1i}y^{(n-1)}(a) + d_{2i}y^{(n-1)}(b) + d_{3i}y^{(n-2)}(a) + d_{4i}y^{(n-2)}(b) + \cdots + \\ +d_{(2n-3)i}y^{(1)}(a) + d_{(2n-2)i}y^{(1)}(b) + d_{(2n-1)i}y(a) + d_{2ni}y(b) = \mu_{1i}, \quad i = \overline{1,n} \end{aligned} \tag{2}$$

are satisfied.

We will consider a division Δ_M of the interval $I = [a, b]$ consisting of $M + 1$ equidistant points: $a = a_0 < a_1 < a_2 < \cdots < a_{M-1} < a_M = b$.

To the equation (1) we attach the following operator:

$$\mathcal{D}(y(x)) = y^{(n)}(x) - F(y^{(n-1)}(x), y^{(n-2)}(x), \cdots, y^{(1)}(x), y(x), x). \tag{3}$$

We denote by $\tilde{y}_i(x)$ an approximate solution of the equation (1) on each subinterval $I = [a_i, a_{i+1}]$.

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By replacing in $\mathcal{D}$ the exact solution $y(x)$ with this approximate solution we obtain the *reminder* on each subinterval:

$$\mathcal{R}_i(x, \tilde{y}_i(x)) = \mathcal{D}(\tilde{y}_i(x)), \qquad x \in [a_i, a_{i+1}]. \tag{4}$$

Definition 1. *We call an* ***ϵ-approximate piecewise polynomial (pwpl) solution*** *of the problem (1 - 2) related to the division Δ_M an approximate polynomial solution $\tilde{y}(x) = \tilde{y}_i(x), x \in I_i$ which satisfies on each subinterval the following relations:*

$$\mathcal{R}_i(x, \tilde{y}_i(x)) < \varepsilon, \quad \varepsilon > 0, \quad i = \overline{0, M}, \tag{5}$$

$$\begin{aligned} d_{1i}\tilde{y}^{(n-1)}(a) + d_{2i}\tilde{y}^{(n-1)}(b) + d_{3i}\tilde{y}^{(n-2)}(a) + d_{4i}\tilde{y}^{(n-2)}(b) + \cdots + \\ d_{(2n-3)i}\tilde{y}^{(1)}(a) + d_{(2n-2)i}\tilde{y}^{(1)}(b) + d_{(2n-1)i}\tilde{y}(a) + d_{2ni}\tilde{y}(b) = \mu_{1i}, \quad i = \overline{1, n} \end{aligned} \tag{6}$$

Definition 2. *We call a* ***week ϵ-approximate piecewise polynomial (pwpl) solution*** *of the problem (1 - 2) an approximate polynomial solution $\tilde{y}(x) = \tilde{y}_i(x), x \in I_i$ which satisfies on each subinterval the relations:*

$$\int_{a_i}^{a_{i+1}} |\mathcal{R}_i(x, \tilde{y}_i(x)) < \varepsilon, \quad \varepsilon > 0, \quad i = \overline{0, M}, \tag{7}$$

$$\begin{aligned} d_{1i}\tilde{y}^{(n-1)}(a) + d_{2i}\tilde{y}^{(n-1)}(b) + d_{3i}\tilde{y}^{(n-2)}(a) + d_{4i}\tilde{y}^{(n-2)}(b) + \cdots + \\ d_{(2n-3)i}\tilde{y}^{(1)}(a) + d_{(2n-2)i}\tilde{y}^{(1)}(b) + d_{(2n-1)i}\tilde{y}(a) + d_{2ni}\tilde{y}(b) = \mu_{1i}, \quad i = \overline{1, n} \end{aligned} \tag{8}$$

Definition 3. *Let*

$$P_{im}(x) = c_{i0} + c_{i1}x + c_{i2}x^2 + \cdots + c_{im}x^m,$$

$c_{ij} \in \mathbb{R}, i = \overline{0, M}, \ j = \overline{0, m}.$

We call the sequence of polynomials $P_{im}(x)$ ***convergent*** *to the solution of the problem (1 - 2) if:*

$$\lim_{m \to \infty} \mathcal{R}_i(x, P_{im}(x)) = 0. \tag{9}$$

According to the Weierstrass Theorem on Polynomial Approximation, it follows that there exists a sequence of polynomials $P_{im}(x)$ which converges to the solution of the equation (1).

It is computed a weak ϵ - approximate pwpl solution, in the sense of the Definition 1, of the type:

$$\tilde{y}_i(x) = \sum_{k=0}^{m} c_{ik}x^k, m > 0 \tag{10}$$

where the coefficients $c_{i0}, c_{i1}, \cdots, c_{im}$ are calculated using the following algorithm:

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1. By substituting the approximate solutions (10) in the equation (1) is obtained the expressions:

$$\mathcal{R}_i(x,\tilde{y}_i(x)) = y_i^{(n)}(x) - F(y_i^{(n-1)}(x), y_i^{(n-2)}(x), \cdots, y_i^{(1)}(x), y_i(x), x)$$
$$i = \overline{0, M-1}, \quad x \in [a_i, a_{i+1}] \tag{11}$$

2. On each interval I_i is attached to the equation (1) the following functional:

$$\mathcal{J}_i(c_{i0}, c_{i1}, c_{i2}, \cdots, c_{im}) = \int_{a_i}^{a_{i+1}} \mathcal{R}_i^2(x,\tilde{y}_i(x))dx \tag{12}$$

3. Is computed the real values $c_{i0}^0, c_{i1}^0, \cdots, c_{im}^0$ as the values which give the minimum of the functionals $\mathcal{J}_i$.

4. With coefficients $c_{i0}^0, \cdots, c_{im}^0$ previously determined is constructed the polynomials:

$$T_{im}(x) = \sum_{k=0}^{m} c_{ik}^0 x^k. \tag{13}$$

Theorem 2.1. *The sequence of polynomials $T_{im}(x)$ from (13) satisfies the property:*

$$\lim_{m\to\infty} \int_{a_i}^{a_i+1} \mathcal{R}_i^2(x, T_{im}(x))dx = 0. \tag{14}$$

Proof: Based on the way the polynomials $T_{im}(t)$ are computed and taking into account the relations (11)-(13), the following inequalities are satisfied:

$$0 \le \int_{a_i}^{a_{i+1}} \mathcal{R}_i^2(x, T_{im}(x))dx \le \int_{a_i}^{a_{i+1}} \mathcal{R}_i^2(x, P_{im}(x))dx, \ \forall m \in \mathbb{N}. \tag{15}$$

It follows that:

$$0 \le \lim_{m\to\infty} \int_{a_i}^{a_{i+1}} \mathcal{R}_i^2(x, T_{im}(x))dx \le \lim_{m\to\infty} \int_{a_i}^{a_{i+1}} \mathcal{R}_i^2(x, P_{im}(x))dx = 0. \tag{16}$$

and:

$$\lim_{m\to\infty} \int_{a_i}^{a_{i+1}} \mathcal{R}_i^2(x, T_{im}(x))dx = 0. \tag{17}$$

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Moreover, $\forall\epsilon > 0,\ \exists m_o \in \mathbb{N},\ m > m_0$ it follows that $T_{im}(x)$ is a *weak ϵ-approximate polynomial solution* of the equation (1) to each interval I_i and the piecewise polynomial:

$$T_m(x) = \begin{cases} T_{1m}(x), x \in [a_0, a_1] \\ T_{2m}(x), x \in [a_1, a_2] \\ \cdots \\ T_{Mm}(x), x \in [a_{M-1}, b] \end{cases} \tag{18}$$

is a *weak ϵ-approximate piecewise polynomial solution* of the equation, on the interval $I = [a, b]$.

3 Application

We consider a lumped system of combined convective–radiative heat transfers. The specific heat coefficient is a linear function of temperature ([7],[10],[11]):

$$c = c_a(1 + \gamma(T - T_a))$$

where γ is a constant and c_a is the specific heat at T_a.

The cooling process of the system is:

$$\rho V c\frac{dT}{d\tau} + hA(T - T_a) + E\sigma A(T^4 - T_s^4) = 0, \qquad T(0) = T_i \tag{19}$$

Performing the changes of variables:
$y = \dfrac{T}{T_i},\ y_a = \dfrac{T_a}{T_i},\ x = \dfrac{\tau(hA)}{\rho V c_a},\ \varepsilon_1 = \gamma T_i, \varepsilon_2 = \dfrac{E\sigma T_i^3}{h},\ y_s = \dfrac{T_s}{T_i}$ and $y_a = y_s = 0$
we obtain the following problem:

$$\begin{gathered} y'(x)(1 + \varepsilon_1 y(x)) + y(x) + \varepsilon_2 y^4(x) = 0, \\ y(0) = 1 \end{gathered} \tag{20}$$

Case 1: $\epsilon_1 = 1, \epsilon_2 = 1$

To verify the effectiveness of the method described in the previous section, an approximate annalytical solution is determined as follows:

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- it divides the interval $I = [0, 1]$ into four subintervals of equal length:
  $0 = a_0 < a_1 < \cdots a_4 = 1$ and is computed on each subinterval $I_i = [a_i, a_{i+1}]$ an approximate solution of the type:

$$\tilde{y}_i(x) = \sum_{k=0}^{m} c_{ik} x^k, \quad m > 0, \quad i = \overline{0,4}.$$

- from initial conditions is determined c_{i0}.
- the corresponding remainders are: $\mathcal{R}_i(x, \tilde{y}_i(x))$.
- the following functionals are computed:

$$\mathcal{J}_i(c_{i1}, c_{i2}, \cdots c_{ik}) = \int_{a_i}^{a_{i+1}} \mathcal{R}_i^2(x, \tilde{y}_i(x))dx$$

Using an mathematical soft for minimizing $\mathcal{J}_i$ it obtaining the PWPLSM solution of the problem (20).

To the interval $I_1 = [0, 0.25]$, we conssider a third degree polynomial:

$$\tilde{y}_1(x) = c_{10} + c_{11}x + c_{12}x^2 + c_{13}x^3$$

and from initial condition $\tilde{y}_1(0) = 0.25$ is determined c_{10}.

The corresponding remainder is:

$$\mathcal{R}_1(x, \tilde{y}_1(x)) = \left(c_{11}x + c_{12}x^2 + c_{13}x^3 + 0.25\right)^4 + \left(c_{11} + 2c_{12}x + 3c_{13}x^2\right)\left(c_{11}x + c_{12}x^2 + c_{13}x^3 + 1.25\right) + c_{11}x + c_{12}x^2 + c_{13}x^3 + 0.25$$

The following functionals is computed: $\mathcal{J}_1(c_{11}, c_{12}, c_{13}) = \int_0^{0.25} \mathcal{R}_1^2(x, \tilde{y}_1(x))dx$

Minimizing $\mathcal{J}_1(c_{11}, c_{12}, c_{13})$ with respect to the real coefficients c_{11}, c_{12}, c_{13}, the PWPLSM solution of the problem (20) on the interval I_1 is:

$$\tilde{y}_1(x) = -0.01156x^3 + 0.0696x^2 - 0.2031x + 0.25$$

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Going through the algorithm for each of the four subintervals I_i, the PWPLSM solution is obtained:

$$\tilde{y}(x) = \begin{cases} -0.01156x^3 + 0.0696x^2 - 0.2031x + 0.25, & x \in [0, 0.25] \\ 0.02558x^4 - 0.08186x^3 + 0.1922x^2 - 0.4573x + 0.6035, & x \in [0.25, 0.5] \\ 0.1646x^4 - 0.6051x^3 + 1.02x^2 - 1.258x + 1.189, & x \in [0.5, 0.75] \\ 0.852x^4 - 3.698x^3 + 6.431x^2 - 5.843x + 3.055, & x \in [0.75, 1] \end{cases}$$

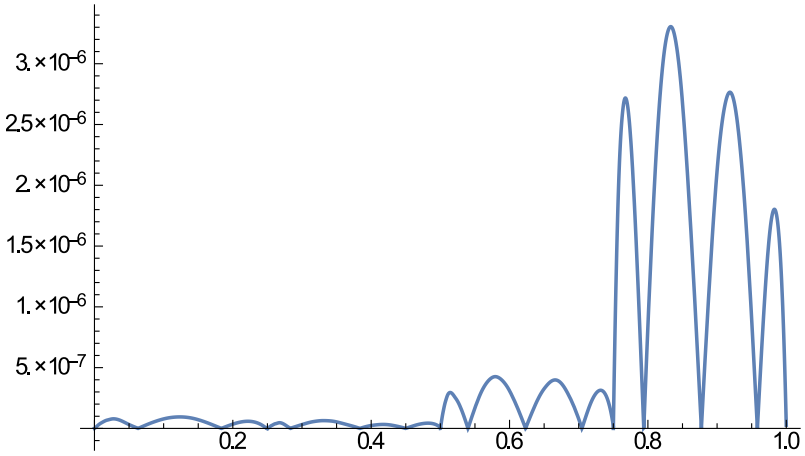


Figure 1: The absolute errors corresponding to $\tilde{y}(x)$ case $\epsilon_1 = 1, \epsilon_2 = 1$ App. 1

Table 1 presents the comparison between the erors obtained by different methods.

Table 1: Comparison of HPM, HAM, SRMM, LSDQ and PWPLSM for $\epsilon_1 = \epsilon_2 = 1$

x	HPM	HAM	$SRMM3^{rd}$ deg	$LSDQ3^{rd}$ deg	$PWPLSM$
0.1	1.209	1.86×10^{-2}	1.704×10^{-3}	5.38×10^{-4}	7.81×10^{-8}
0.2	7.24×10^{-1}	5.82×10^{-3}	4.66×10^{-4}	1.403×10^{-3}	3.49×10^{-8}
0.3	4.27×10^{-1}	6.18×10^{-3}	1.19×10^{-3}	3.37×10^{-3}	3.16×10^{-8}
0.4	2.47×10^{-1}	1.709×10^{-2}	2.12×10^{-3}	4.302×10^{-3}	2.36×10^{-8}
0.5	1.38×10^{-1}	2.68×10^{-2}	1.99×10^{-3}	3.89×10^{-3}	8.45×10^{-9}
0.6	7.49×10^{-2}	3.52×10^{-2}	9.44×10^{-4}	2.38×10^{-3}	3.09×10^{-7}
0.7	3.81×10^{-2}	4.25×10^{-2}	5.48×10^{-4}	3.06×10^{-4}	7.73×10^{-8}
0.8	1.78×10^{-2}	4.85×10^{-2}	1.78×10^{-3}	1.57×10^{-3}	8.41×10^{-7}
0.9	7.303×10^{-2}	5.34×10^{-2}	1.93×10^{-3}	2.35×10^{-3}	2.11×10^{-6}

We will compare the absolute errors for approximate PWPLSM solutions with previous solutions: HPM obtained by Ganji et all in ([10]), HAM obtained by

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Damiary et all in ([12]) and $SRMM$ obtained by Caruntu and Bota in ([7]), obtained by Pasca et all in ([9]). Since the problem (20) does not have a known exact solution, we computed for each approximate solution the relative error as the difference (in absolute value) between the approximate solution and the numerical solution given by the Wolfram Mathematica software.

Better errors are observed in the case of the PWPLSM solution, even if the approximation was made with a polynomial of third degree.

Case 2: $\epsilon_1 = 1, \epsilon_2 = 0$

Using PWPLSM we computed the following piecewise polynomial approximate solution of equation (20):

$$\tilde{y}(x) = \begin{cases} 0.0613117x^2 - 0.199774x + 0.25, & x \in [0, 0.25] \\ 0.0738664x^2 - 0.370244x + 0.587944, & x \in [0.25, 0.5] \\ 0.0712664x^2 - 0.499941x + 0.982154, & x \in [0.5, 0.75] \\ 1.41134 - 0.596745x + 0.064394x^2, & x \in [0.75, 1] \end{cases}$$

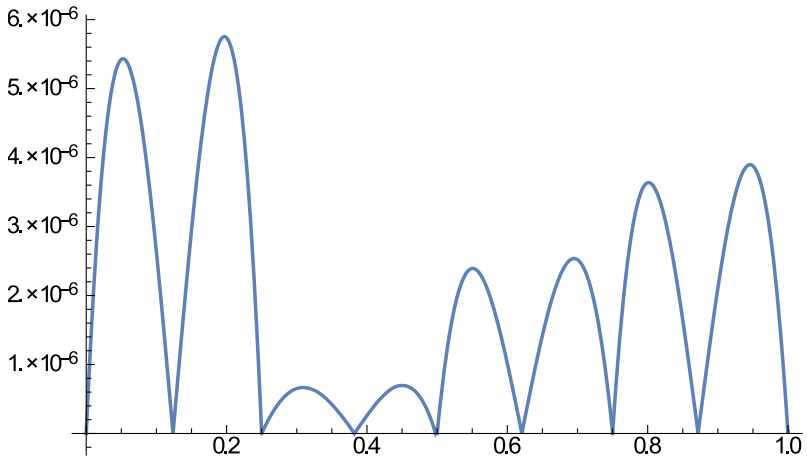


Figure 2: The absolute errors corresponding to $\tilde{y}(x)$ case $\epsilon_1 = 1, \epsilon_2 = 0$ App. 1

As in Case 1, we compare in Table 2 the erors from PWPLSM solutions with previous one: HPM obtained by Ganji in ([13]), HAM obtained by Abbasbandy in ([14]), $OHAM$ by Marinca and Herisanu in ([4]), $SRMM$ by Caruntu and Bota in ([7]) and Pasca et all in ([9]).

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Table 2: Comparison erors of HPM, HAM, OHAM, SRMM, LSDQ and PWPLSM for $\epsilon_1 = 1, \epsilon_2 = 0$ in Application 1

x	HPM	HAM	$OHAM$	$SRMM2^{nd}$ deg	$LSDQ2^{nd}$ deg	$PWPLSM$
0.1	3.35×10^{-2}	3.67×10^{-5}	3.708×10^{-2}	1.56×10^{-3}	1.07×10^{-4}	2.63×10^{-4}
0.2	4.345×10^{-2}	1.954×10^{-3}	5.366×10^{-2}	8.024×10^{-5}	2.67×10^{-4}	5.73×10^{-6}
0.3	4.029×10^{-2}	4.091×10^{-3}	5.658×10^{-2}	1.138×10^{-3}	5.698×10^{-5}	6.51×10^{-7}
0.4	3.071×10^{-2}	5.415×10^{-3}	5.113×10^{-2}	1.669×10^{-3}	4.325×10^{-5}	2.58×10^{-7}
0.5	1.886×10^{-2}	5.541×10^{-3}	4.118×10^{-2}	1.730×10^{-3}	1.571×10^{-4}	7.42×10^{-8}
0.6	7.170×10^{-3}	4.432×10^{-3}	2.942×10^{-2}	1.379×10^{-3}	2.600×10^{-4}	1.01×10^{-6}
0.7	3.062×10^{-3}	2.243×10^{-3}	1.762×10^{-2}	6.736×10^{-4}	3.292×10^{-4}	2.51×10^{-6}
0.8	1.125×10^{-2}	7.827×10^{-4}	6.855×10^{-3}	3.304×10^{-4}	3.449×10^{-4}	3.63×10^{-6}
0.9	1.726×10^{-2}	4.372×10^{-3}	2.320×10^{-3}	1.575×10^{-3}	2.903×10^{-4}	2.11×10^{-6}

Case 3: $\epsilon_1 = 0, \epsilon_2 = 1$

In this case, the solution was calculated keeping, as in the other two cases, four equidistant subintervals of the interval $I = [a.b]$. On each subinterval $I_i = [a_i, a_{+i+1}]$ the approximate solution $\tilde{y}(x)$ is a third degree polynomial, the four coefficients were calculated to determine the approximate polynomial $\tilde{y}_i$.

For this case is computed the following piecewise polynomial least squares annalytical approximate solution:

$$\tilde{y}(x) = \begin{cases} 0.25 - 0.253847x + 0.133416x^2 - 0.0453598x^3, & x \in [0, 0.25] \\ 0.669223 - 0.806281x + 0.574651x^2 - 0.22838x^3, & x \in [0.25, 0.5] \\ 1.73516 - 3.16098x + 2.94193x^2 - 1.12119x^3, & x \in [0.5, 0.75] \\ 6.39624 - 14.8875x + 13.5242x^2 - 4.35673x^3, & x \in [0.75, 1] \end{cases}$$

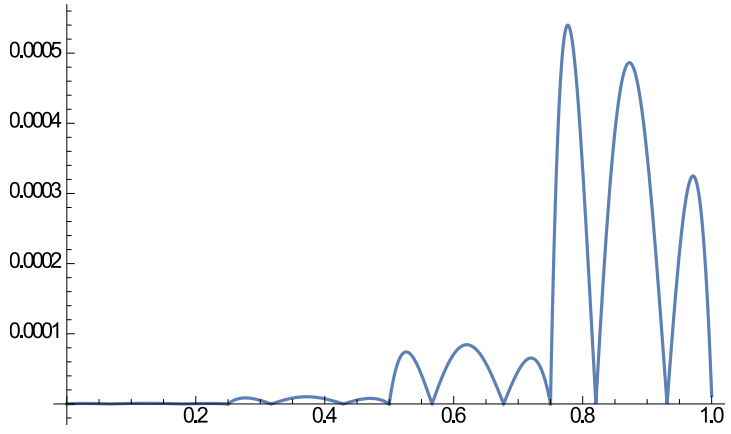


Figure 3: The absolute errors corresponding to $\tilde{y}(x)$ case $\epsilon_1 = 0, \epsilon_2 = 1$ App. 1

In Table 3 we present the comparison between the errors of solutions obtained

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by PWPLSM and the erors of solution obtained by Rajabi et all in ([15]) using HPM, Domairry et all in ([12]) using HAM, Caruntu and Bota in ([7]) using $SRMM$ and Pasca et all usind LSDQM in ([9]).

Table 3: Comparison of HPM, HAM, SRMM, LSDQ and PWPLSM erors for $\epsilon_1 = 0, \epsilon_2 = 1$ in Application 1

t	HPM	HAM	$SRMM5^{th}$ deg	$LSDQ5^{nd}$ deg	$PWPSM$
0.1	2.24×10^{-3}	1.18×10^{-5}	6.08×10^{-5}	2.52×10^{-3}	7.18×10^{-7}
0.2	9.47×10^{-3}	8.72×10^{-4}	1.25×10^{-3}	3.51×10^{-3}	4.47×10^{-7}
0.3	1.79×10^{-2}	1.25×10^{-3}	3.12×10^{-3}	1.40×10^{-3}	5.05×10^{-6}
0.4	2.51×10^{-2}	1.54×10^{-2}	9.67×10^{-3}	7.42×10^{-3}	7.31×10^{-6}
0.5	3.004×10^{-2}	2.09×10^{-2}	1.12×10^{-3}	1.06×10^{-3}	1.25×10^{-9}
0.6	3.27×10^{-2}	3.04×10^{-2}	1.78×10^{-4}	2.87×10^{-3}	7.08×10^{-5}
0.7	3.37×10^{-2}	4.35×10^{-2}	8.14×10^{-4}	1.83×10^{-4}	4.69×10^{-5}
0.8	3.35×10^{-2}	5.95×10^{-2}	7.43×10^{-3}	1.94×10^{-3}	3.34×10^{-4}
0.9	3.23×10^{-2}	7.72×10^{-2}	3.59×10^{-3}	2.69×10^{-3}	3.54×10^{-4}

4 Conclusions

In the present paper we obtain analytical approximate solutions for nonlinear heat transfer problems using the recently introduced Piecewise Polynomial Least Squares Method. Using the Piecewise Polynomial Least Squares Method one obtains the analytical solution of the problem, not only numerical solutions, fact which demonstrates the usefulness of the method (PWPLSM).

The application presented clearly illustrate the accuracy of the method. The very good errors (smaller than those existing in the literature) that can be observed in each of the three tables, were calculated starting from approximate solutions, polynomial of not very high degree. If in ([7]) using SRMM and in ([9]) with LSDQ, the approximation polynomial had five degree, with this new method we only used third degree polynomial and the approximate analytical solution found being closer to the exact solution. The degree of the approximation polynomial being lower, fewer coefficients are calculated that minimize the functional $\mathcal{J}$, which means that the required resources are also fewer.

For all these reasons, it appears that the proposed method is useful and easy to use in practice.

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