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MAJORIZATION CRITERIA FOR POLYNOMIAL STABILITY AND INSTABILITY OF EVOLUTION OPERATORS

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Abstract

The aim of this paper is to present majorization criteria for the uniform polynomial stability, respectively for the uniform polynomial instability of evolution operators in Banach spaces. In this sense we establish connections with the exponential case and we give two characterization theorems for the concepts mentioned above. [1]

Keywords and phrases: *majorization criterion, uniform polynomial stability, evolution operator.*

1 Introduction

One of the most important results in the stability theory of evolution operators was obtained by Datko (see [4]) who gave an integral characterization of the uniform exponential stability concept. This paper served as a starting point for many works in which the authors obtained important results concerning the exponential asymptotic behaviors in Banach spaces (see [8, 10, 11, 12]).

Over the last years, the notion of polynomial asymptotic behaviors has been very well developed. It has been considered in the works of Barreira and Valls [2] for evolution operators and Bento and Silva [3] for discrete-time systems. Moreover, important contributions to the study of polynomial stability and instability has been made (see [5, 7, 9]).

[1] MSC (2010): 34D05, 34D20

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The present paper will focus on the uniform polynomial stability and uniform polynomial instability of evolution operators. More precisely, we will give majorization criteria for the concepts mentioned above, by extending the techniques used in the exponential case (see [6]) to the polynomial case.

2 Preliminary notions

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators acting on X. The norms on X and on $\mathcal{B}(X)$ will be denoted by $\|.\|$. The identity operator on X is denoted by I. Also, we consider the sets

$$\Delta = \{(t,s) \in I\!R_+^2 : t \geq s\}, \qquad T = \{(t,s,t_0) \in I\!R_+^3 : t \geq s \geq t_0\}$$

Definition 2.1. An application $U : \Delta \to \mathcal{B}(X)$ is said to be an *evolution operator* on X if

(e_1) $U(t,t) = I$ for every $t \geq 0$

(e_2) $U(t,s)U(s,t_0) = U(t,t_0)$ for all $(t,s,t_0) \in T$.

Definition 2.2. The evolution operator $U : \Delta \to \mathcal{B}(X)$ is *uniformly exponentially stable (u.e.s.)*, if there are $N \geq 1$ and $\nu > 0$ such that:

$$\|U(t,s)x\| \leq Ne^{-\nu(t-s)}\|x\|$$

for all $(t,s,x) \in \Delta \times X$.

Definition 2.3. The evolution operator $U : \Delta \to \mathcal{B}(X)$ is *uniformly polynomially stable (u.p.s.)*, if there are $N \geq 1$ and $\nu > 0$ such that:

$$(t+1)^\nu \|U(t,s)x\| \leq N(s+1)^\nu \|x\|$$

for all $(t,s,x) \in \Delta \times X$.

Definition 2.4. The evolution operator $U : \Delta \to \mathcal{B}(X)$ *has a uniform polynomial growth (u.p.g.)* if there are $M \geq 1$ and $\omega > 0$ such that

$$(s+1)^\omega \|U(t,s)x\| \leq M(t+1)^\omega \|x\|$$

for all $(t,s,x) \in \Delta \times X$.

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Remark 2.5. *It is obvious that*

$$u.p.s. \Rightarrow u.s. \Rightarrow u.p.g.$$

Definition 2.6. The evolution operator $U : \Delta \to \mathcal{B}(X)$ is *uniformly exponentially instable (u.e.is.)*, if there are $N \geq 1$ and $\nu > 0$ such that:

$$N\|U(t,s)x\| \geq e^{\nu(t-s)}\|x\|$$

for all $(t,s,x) \in \Delta \times X$.

Definition 2.7. The evolution operator $U : \Delta \to \mathcal{B}(X)$ is *uniformly polynomially instable (u.p.is.)*, if there are $N \geq 1$ and $\nu > 0$ such that:

$$(t+1)^{\nu}\|x\| \leq N(s+1)^{\nu}\|U(t,s)x\|$$

for all $(t,s,x) \in \Delta \times X$.

Definition 2.8. The evolution operator $U : \Delta \to \mathcal{B}(X)$ *has a uniform polynomial decay (u.p.d.)* if there are $M \geq 1$ and $\omega > 0$ such that

$$(s+1)^{\omega}\|x\| \leq M(t+1)^{\omega}\|U(t,s)x\|$$

for all $(t,s,x) \in \Delta \times X$.

Remark 2.9. *It is easy to see that*

$$u.p.is. \Rightarrow u.is. \Rightarrow u.p.d.$$

Theorem 2.10. *Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform exponential growth. Then, U is uniformly exponentially stable if and only if there exists a nondecreasing application*

$$\varphi : [1,\infty) \to \mathbb{R}_+ \text{ with } \lim_{t\to\infty} \varphi(t) = \infty \text{ and}$$

$$\varphi(t-s)\|U(t,t_0)x_0\| \leq \|U(s,t_0)x_0\|,$$

for all $(t,s,t_0,x_0) \in T \times X$.

Proof. See [6]. □

Theorem 2.11. *Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform exponential decay. Then U is uniform exponentially instable if and only if there exists a nondecreasing function $\varphi : [1,\infty) \to \mathbb{R}_+$ with $\lim_{t\to\infty} \varphi(t) = \infty$ and*

$$\varphi(t-s)\|U(s,t_0)x_0\| \leq \|U(t,t_0)x_0\|,$$

for all $(t,s,t_0,x_0) \in T \times X$.

Proof. See [6]. □

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3 Main results

Let us consider the evolution operators

$U_1, U_2 : \Delta \to \mathcal{B}(X)\ U_1(t,s) = U(\ln(t+1), \ln(s+1)),\ U_2(t,s) = U(e^t-1, e^s-1),$

for all $(t,s) \in \Delta \times X$.

Proposition 3.1. *The evolution operator U_1 is uniformly polynomially stable if and only if U is uniformly exponentially stable.*

Indeed, U u.e.s. is equivalent with

$$\|U_1(t,s)\| = \|U(\ln(t+1), \ln(s+1))\| \leq Ne^{-\nu(\ln(t+1)-\ln(s+1))} = N\left(\frac{t+1}{s+1}\right)^{-\nu}$$

which means U_1 u.p.s.

Proposition 3.2. *The evolution operator U is uniformly polynomially stable if and only if U_2 is uniformly exponentially stable.*

Proof. Necessity We suppose that U is u.p.s. Then

$$\|U_2(t,s)\| = \|U(e^t-1, e^s-1)\| \leq N\left(\frac{e^s}{e^t}\right)^{\nu} = Ne^{-\nu(t-s)},$$

which implies U_2 u.e.s.
Sufficiency Let U_2 be u.e.s. Then

$$\|U_2(t,s)\| = \|U(e^t-1, e^s-1)\| \leq Ne^{-\nu(\ln(1+u)-\ln(1+v))} = Ne^{-\nu\ln\frac{1+u}{1+v}} =$$
$$= Ne^{\ln\left(\frac{1+u}{1+v}\right)^{-\nu}} = N\left(\frac{1+u}{1+v}\right)^{-\nu},$$

which implies U u.p.s. □

Proposition 3.3. *The evolution operator U_1 is uniformly polynomially instable if and only if U is uniformly exponentially instable.*

It results immediately, because U u.e.is. is equivalent with

$$N\|U_1(t,s)\| = N\|U(\ln(t+1), \ln(s+1))x\| \geq e^{\nu(\ln(t+1)-\ln(s+1))}\|x\| =$$
$$= e^{\nu\ln\frac{t+1}{s+1}}\|x\| = \left(\frac{t+1}{s+1}\right)^{\nu}\|x\|,$$

so U_1 is u.p.is.

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Proposition 3.4. *The evolution operator U is uniformly polynomially instable if and only if U_2 is uniformly exponentially instable.*

Proof. If U u.p.is. then

$$N\|U_2(t,s)x\| = N\|U(e^t-1,e^s-1)x\| \geq \left(\frac{e^t}{e^s}\right)^{\nu}\|x\| = e^{\nu(t-s)}\|x\|,$$

which implies that U_2 is u.e.is.
Conversely if U_2 u.e.is., then we have

$$N_2\|U_2(t,s)x\| = N_2\|U(e^t-1,e^s-1)x\| = N_2\|U(u,v)x\| \geq$$
$$\geq e^{\nu(\ln(1+u)-\ln(1+v))}\|x\| = e^{\nu\ln\frac{1+u}{1+v}}\|x\| = \left(\frac{1+u}{1+v}\right)^{\nu}\|x\|,$$

which involves that U is u.p.is. □

The main result of the paper is the next theorem which is an majorization criterion for the concept of uniform polynomial stability of an evolution operator.

Theorem 3.5. *Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform polynomial growth. Then, U is uniformly polynomially stable if and only if there exists a nondecreasing application*

$$\varphi : [1,\infty) \to \mathbb{R}_+ \text{ with } \lim_{t\to\infty}\varphi(t) = \infty \text{ and}$$

$$\varphi\left(\frac{t+1}{s+1}\right)\|U(t,t_0)x_0\| \leq \|U(s,t_0)x_0\|,$$

for all $(t,s,t_0,x_0) \in T \times X$.

Proof. Necessity.
We suppose that U is u.p.s., which implies using Remark 2.5 and Proposition 3.2 that U has u.p.g. and U_2 is u.e.s., namely U_2 has u.e.g. Then, from the majorization criterion for the uniform exponential stability (see Theorem 2.10) we have that there exists a nondecreasing function $\varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty}\varphi_2(t) = \infty$ and

$$\varphi_2(u-v)\|U_2(u,w)x_0\| \leq \|U_2(v,w)x_0\|, (*)$$

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for all $(u, v, w, x_0) \in T \times X$.
Let $u = \ln(t+1),\ v = \ln(s+1),\ w = \ln(t_0+1)$ which implies

$$t = e^u - 1,\ s = e^v - 1,\ t_0 = e^w - 1.$$

Then $\varphi_2(u-v) = \varphi_2\left(\ln\left(\frac{t+1}{s+1}\right)\right) = \varphi\left(\frac{t+1}{s+1}\right)$

$$U(t, t_0) = U(e^u - 1, e^w - 1) = U_2(u, w)$$
$$U(s, t_0) = U(e^v - 1, e^w - 1) = U_2(v, w)$$

If we replace in relation $(*)$ we obtain

$$\varphi\left(\frac{t+1}{s+1}\right)\|U(t, t_0)x_0\| \leq \|U(s, t_0)x_0\|,$$

which is equivalent with the conclusion.
Sufficiency. We suppose that there exists a nondecreasing function

$$\varphi : [1, \infty) \to I\!R_+ \text{ with } \lim_{t\to\infty} \varphi(t) = \infty$$

and

$$\varphi\left(\frac{t+1}{s+1}\right)\|U(t, t_0)x_0\| \leq \|U(s, t_0)x_0\|(**)$$

for all $(t, s, t_0, x_0) \in T \times X$. We have to prove that U is u.p.s, which is equivalent with U_2 is u.e.s.
Let $\varphi_2(x) = \varphi(e^x)$. Let $t = e^u - 1,\ s = e^v - 1,\ t_0 = e^w - 1$, that implies $u = \ln(t+1),\ v = \ln(s+1),\ w = \ln(t_0+1)$. Then

$$\varphi\left(\frac{t+1}{s+1}\right) = \varphi\left(e^{\ln\frac{t+1}{s+1}}\right) = \varphi(e^{\ln(t+1)-\ln(s+1)}) = \varphi(e^{u-v}) = \varphi_2(u-v)$$

$$\|U(t, t_0)x_0\| = \|U(e^u - 1, e^w - 1)x_0\| = \|U_2(u, w)x_0\|$$
$$\|U(s, t_0)x_0\| = \|U(e^v - 1, e^w - 1)x_0\| = \|U_2(v, w)x_0\|$$

If we replace in $(**)$ we obtain

$$\varphi_2(u-v)\|U_2(u, w)x_0\| \leq \|U_2(v, w)x_0\|,$$

so U_2 is u.e.s. □

The following theorem is a majorization criterion for the instability concept.

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Theorem 3.6. *Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform polynomial decay. Then U is uniform polynomially instable if and only if there exists a nondecreasing function*

$$\varphi : [1, \infty) \to \mathbb{R}_+ \text{ with } \lim_{t\to\infty} \varphi(t) = \infty \text{ and}$$

$$\varphi\left(\frac{t+1}{s+1}\right) \|U(s, t_0)x_0\| \leq \|U(t, t_0)x_0\|,$$

for all $(t, s, t_0, x_0) \in T \times X$.

Proof. Necessity.
We suppose that U is u.p.is. From Remark 2.9 and Proposition 3.4 we have that U has u.p.d. and U_2 is u.e.is., which implies that U_2 has u.e.d. Then, from the majorization criterion for the uniform exponential instability (see Theorem 2.11) we have that there exists a nondecreasing application $\varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} \varphi_2(t) = \infty$ şi

$$\varphi_2(u - v)\|U_2(v, w)x_0\| \leq \|U_2(u, w)x_0\|, (*)$$

for all $(u, v, w, x_0) \in T \times X$.

$$\forall u \in \mathbb{R}_+ \exists t \in \mathbb{R}_+ : u = \ln(t + 1) \Rightarrow e^u = t + 1 \Rightarrow t = e^u - 1$$

$$\forall v \in \mathbb{R}_+ \exists s \in \mathbb{R}_+ : v = \ln(s + 1) \Rightarrow e^v = s + 1 \Rightarrow s = e^v - 1$$

$$\forall w \in \mathbb{R}_+ \exists t_0 \in \mathbb{R}_+ : w = \ln(t_0 + 1) \Rightarrow e^w = t_0 + 1 \Rightarrow t_0 = e^w - 1$$

Because $u \geq v \geq w$, we obtain $t \geq s \geq t_0$. We compute
$\varphi_2(u - v) = \varphi_2\left(\ln\left(\frac{t+1}{s+1}\right)\right) = \varphi\left(\frac{t+1}{s+1}\right)$, where $\varphi = \varphi_2 \circ \ln$

$$\|U_2(u, w)x_0\| = \|U(e^u - 1, e^w - 1)x_0\| = \|U(t, t_0)x_0\|$$
$$\|U_2(v, w)x_0\| = \|U(e^v - 1, e^w - 1)x_0\| = \|U(s, t_0)x_0\|$$

If we replace in $(*)$ we obtain

$$\varphi\left(\frac{t+1}{s+1}\right) \|U(s, t_0)x_0\| \leq \|U(t, t_0)x_0\|,$$

so the necessity is proved.

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Sufficiency.
We suppose that there exists a nondecreasing function

$$\varphi : [1, \infty) \to I\!R_+$$

with $\lim\limits_{t\to\infty} \varphi(t) = \infty$ and

$$\varphi\left(\frac{t+1}{s+1}\right) \|U(s, t_0)x_0\| \leq \|U(t, t_0)x_0\| (**)$$

for all $(t, s, t_0, x_0) \in T \times X$. We need to show that U is u.p.is, namely U_2 is u.e.is.
Let $t = e^u - 1,\ s = e^v - 1,\ t_0 = e^w - 1$, which implies

$$u = \ln(t+1),\ v = \ln(s+1),\ w = \ln(t_0+1).$$

Then

$$\varphi\left(\frac{t+1}{s+1}\right) = \varphi\left(e^{\ln\frac{t+1}{s+1}}\right) = \varphi(e^{\ln(t+1)-\ln(s+1)}) = \varphi(e^{u-v}) = \varphi_2(u-v),$$

where $\varphi_2 = \varphi \circ \exp$.

$$\|U(t, t_0)x_0\| = \|U(e^u - 1, e^w - 1)x_0\| = \|U_2(u, w)x_0\|$$

$$\|U(s, t_0)x_0\| = \|U(e^v - 1, e^w - 1)x_0\| = \|U_2(v, w)x_0\|$$

If we replace in the relation $(**)$ we obtain the conclusion. □

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