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A NOTE ON A WELL-POSEDNESS RESULT

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Abstract

We study the well-posedness in the smooth category, for a class of Euler equations. The results obtained by Escher, Bauer and Kolev are extended, in the case $d = 1$, for the class of pseudo-differential operators with x-compactly supported symbols. [1]

Keywords and phrases: *Euler equations, Nash-Moser theorem, elliptic pseudo-differential operators.*

1 Introduction

In two seminal articles V. Arnold [1], respectively D. Ebin, J.E. Marsden [4] created an alternative, to the Nash-Moser schemes, for proving well-posedness results in the smooth category. This alternative is called nowadays the geometric method in hydrodynamics, according to [6]. The main idea is to recast an equation as a geodesic equation, corresponding to a right-invariant metric, on an infinite dimensional manifold, usually a group of diffeomorphisms. The spray equation, via a "no loss, nor gain in spatial regularity" result, will give the possibility of obtaining a Cauchy-Lipschitz type result on Fréchet spaces. In order to prove the smoothness of the spray, on Banach approximations of the tangent bundle, one needs a boundedness result for some multi-linear commutators. In this note we present the boundedness result corresponding to a pseudo-differential operator on $\mathbb{R}$, with an x-compactly supported symbol. This leads to an extension, to inertia operators of pseudo-differential type, of the results presented in [2], in the particular case $d = 1$.

2 The Geometric Approach for Well-posedness

In the sequel we exemplify the geometric method for well-posedness. First of all, let us consider the Lie group $\mathrm{Diff}_{H^\infty}(\mathbb{R})$ defined as:

$$\mathrm{Diff}_{H^\infty}(\mathbb{R}) = \{id + u : \ u \in H^\infty(\mathbb{R}) \text{ and } u' > -1\},$$

[1] MSC (2010): 35Q35, 35S05, 46A61, 58D05

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where:

$$H^\infty(\mathbb{R}) = \bigcap_{q\geq 0} H^q(\mathbb{R}).$$

We will also introduce the Hilbert manifolds:

$$\mathcal{D}^q(\mathbb{R}) = \{id + u :\ u \in H^q(\mathbb{R}) \text{ and } u' > -1\}.$$

These are actually topological groups for $q > \frac{3}{2}$, but not Lie groups.

We introduce further *the class* $\mathcal{E}^r$ of pseudo-differential operators A as the class of pseudo-differential operators on $\mathbb{R}$, included in Hörmander's class $OpS^r_{1,0}$, that are also elliptic, invertible and hermitian, i.e:

$$\sigma_A(x,-\xi) = \overline{\sigma_A(x,\xi)}, \quad x,\xi \in \mathbb{R},$$

where σ_A is the symbol of A. A pseudo-differential operator with a hermitian symbol will send real-valued functions into real-valued functions.

We build inner products on $H^\infty(\mathbb{R})$ of the form:

$$\langle u, v\rangle = \int_{\mathbb{R}} Au \cdot v \; dx,$$

where A is a pseudo-differential operator of class $\mathcal{E}^r$. Using right translations,

$$R_\varphi(v) := v \circ \varphi,$$

one can obtain a metric on $\mathrm{Diff}_{H^\infty}(\mathbb{R})$ via:

$$G_\varphi(u_\varphi, v_\varphi) := \langle R_{\varphi^{-1}}(u_\varphi), R_{\varphi^{-1}}(v_\varphi)\rangle = \int_{\mathbb{R}} A_\varphi u_\varphi \cdot v_\varphi \cdot \varphi' \, dx, \tag{2.1}$$

where $u_\varphi, v_\varphi \in T_\varphi \mathrm{Diff}_{H^\infty}(\mathbb{R})$ and A_φ is the twisted operator defined as $A_\varphi = R_\varphi \circ A \circ R_\varphi^{-1}$. A geodesic corresponding to the metric G on $\mathrm{Diff}_{H^\infty}(\mathbb{R})$ is an extremal curve $\varphi(t)$ of the energy functional:

$$E(\varphi) := \frac{1}{2}\int_0^1 G_\varphi(\dot\varphi(t), \dot\varphi(t))\, dt.$$

If we denote by $u(t) := R_{\varphi^{-1}(t)}\dot\varphi(t)$ the *Eulerian velocity* of the geodesic curve $\varphi(t)$, then according to [1] the curve $\varphi(t)$ is a geodesic if and only if $u(t)$ is a solution of the *Euler-Poincaré equation*:

$$m_t + 2m \cdot u_x + m_x \cdot u = 0, \quad m := Au\,. \tag{2.2}$$

Since A is invertible, the Euler-Poincaré equation (2.2) can be written as

$$u_t = -A^{-1}\left\{2(Au)\cdot u_x + (Au)_x \cdot u\right\}, \tag{2.3}$$

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and we call this equation the *Euler–Arnold equation* on $\mathrm{Diff}_{H^\infty}(\mathbb{R})$ for the *inertia operator* A.

On Banach scales $H^q(\mathbb{R})$, the Euler-Arnold equation is of order 1, since if $u \in H^q(\mathbb{R})$ then $A^{-1}\{u \cdot (Au)_x\} \in H^{q-1}(\mathbb{R})$. This derivative loss leads in general to a Nash-Moser approach. In the case investigated in this article the well-posedness in the smooth category can not be investigated with a Nash-Moser scheme, since the space $H^\infty(\mathbb{R})$ is not a tame Fréchet space in the sense of [7]. The lack of suitable interpolation inequalities for the space $C^\infty([-1,1], H^\infty(\mathbb{R}))$ is also a major obstacle in applying a Nash-Moser approach. Thus, in the case presented here, the geometric approach seem to have no contenders.

The geometric approach is based on the following phenomenon: the spray equation (Lagrangian coordinates) behaves better than the Euler-Arnold equation (Eulerian coordinates). One can interpret the spray equation as an ODE on the Hilbert approximations $T\mathcal{D}^q(\mathbb{R})$ of the tangent bundle $T\mathrm{Diff}_{H^\infty}(\mathbb{R})$. We give more details below.

Let φ be the flow of the time dependent vector field u and let $v = \varphi_t$. Then

$$v_t = (u_t + uu_x) \circ \varphi$$

and u solves the Euler equation if and only if (φ, v) is a solution of:

$$\begin{cases} \varphi_t = v \\ v_t = S_\varphi(v) \end{cases} \tag{2.4}$$

where

$$S_\varphi(v) := (R_\varphi \circ S \circ R_{\varphi^{-1}})(v)$$

and

$$S(u) := A^{-1}\{[A,u]u_x - 2(Au)_x\}.$$

It is worth mentioning that if A is a pseudo-differential operator in the class $\mathcal{E}^r$, with $r \geq 1$, then the operator $S(u)$ is of order 0 because the commutator $[A,u]$ is of order less than or equal to $r-1$. That is the essential remark regarding the evolution equation (2.4).

We call the second order vector field F on $\mathrm{Diff}_{H^\infty}(\mathbb{R})$:

$$F : (\varphi, v) \to (\varphi, v, v, S_\varphi(v))$$

the *geodesic spray*, since locally it corresponds to the spray introduced in [9], in the context of Banach manifolds.

The strategy is to prove that the geodesic spray F defined above extends to a smooth mapping on the tangent bundle $T\mathcal{D}^q(\mathbb{R})$. Then one can apply the Cauchy-Lipschitz theorem and there will exist for each $(\varphi_0, v_0) \in T\mathcal{D}^q(\mathbb{R})$ a unique non-extendable solution $(\varphi, v) \in C^\infty\left(J_q(\varphi_0, v_0), T\mathcal{D}^q(\mathbb{R})\right)$ of the Cauchy problem (2.4) with $\varphi(0) = \varphi_0$ and $v(0) = v_0$. Here $J_q(\varphi_0, v_0)$ is the maximal interval of existence. The "No loss, nor Gain

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result" [4] states that $J_{q+1}(\varphi_0, v_0) = J_q(\varphi_0, v_0)$ for an initial data $(\varphi_0, v_0) \in T\mathcal{D}^q(\mathbb{R})$, when $q > r + \frac{1}{2}$. Thus the possibility of having $\bigcap_q J_q(\varphi_0, v_0) = \{0\}$ is excluded and the solution extends to a smooth one:

Theorem 2.1 *Let A be a pseudo-differential operator with an x-compactly symbol, of class $\mathcal{E}^r$ with $r \geq 1$. Then, given any $(\varphi_0, v_0) \in T\mathrm{Diff}_{H^\infty}(\mathbb{R})$, there exists a unique non-extendable geodesic $(\varphi, v) \in C^\infty(J, T\mathrm{Diff}_{H^\infty}(\mathbb{R}))$ on the maximal interval of existence J, which is open and contains* 0.

Corollary 2.2 *The corresponding Euler-Arnold equation* (2.3) *has for any initial data $u_0 \in H^\infty(\mathbb{R})$ a unique non-extendable solution $u \in C^\infty(J, H^\infty(\mathbb{R}))$. The maximal interval of existence J is open and contains* 0. *Moreover, the solution depends smoothly on u_0.*

In the particular case $A = I - D_x^2$ the Euler-Poincaré equation (2.2) becomes the Camassa-Holm equation:

$$u_t - u_{txx} = 2u_x u_{xx} + uu_{xxx} - 3uu_x \ .$$

For this equation it has been proved in [8] that the solution map can not be more than continuous, for initial data $u_0 \in H^s(\mathbb{R})$. Hence, according to the above corollary, a restriction to smooth initial data $u_0 \in H^\infty(\mathbb{R})$ furnishes smooth dependence, not only continuous.

Following [2] or [5] the major argument used in the proof of Theorem 2.1 relies on proving the smoothness of the twisted operator $\varphi \to A_\varphi$ at the level of $\mathcal{D}^q(\mathbb{R})$. It is worth mentioning that the tangent bundle $T\mathcal{D}^q(\mathbb{R})$ is trivial, hence $T\mathcal{D}^q(\mathbb{R}) \cong \mathcal{D}^q(\mathbb{R}) \times H^q(\mathbb{R})$. The following proposition, together with the "no gain, nor loss" result, will demonstrate the statement of Theorem 2.1, (see [2], for details):

Proposition 2.3 *Let $A \in \mathcal{E}^r$ with an x-compactly supported symbol. Then the mapping:*

$$\varphi \mapsto A_\varphi := R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q \to \mathcal{L}(H^q(\mathbb{R}), H^{q-r}(\mathbb{R}))$$

is smooth for $q > r + 1/2$ and $r \geq 1$.

It has been observed in [2],[5] that the n-th partial Gâteaux derivative of $\varphi \mapsto A_\varphi$, for smooth arguments, is given by:

$$\partial_\varphi^n A_\varphi(v, \delta\varphi_1, \ldots, \delta\varphi_n) = R_\varphi A_n R_\varphi^{-1}(v, \delta\varphi_1, \ldots, \delta\varphi_n),$$

where:

$$A_n := \partial_{\mathrm{id}}^n A_\varphi \in \mathcal{L}^{n+1}(H^\infty(\mathbb{R}), H^\infty(\mathbb{R}))$$

is the $(n+1)$-linear operator defined inductively by $A_0 = A$, and:

$$\begin{aligned} A_{n+1}(u_0, u_1, \ldots, u_{n+1}) &= u_{n+1} D\left(A_n(u_0, u_1, \ldots, u_n)\right) \\ &\quad - \sum_{k=0}^{n} A_n(u_0, u_1, \ldots, u_{n+1} D(u_k), \ldots, u_n), \end{aligned} \tag{2.5}$$

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where $D = \frac{d}{dx}$. This observation reduces the problem (see [5, Theorem 3.4]) to showing that each A_n extends to a bounded $(n+1)$-linear operator from $H^q(\mathbb{R})$ to $H^{q-r}(\mathbb{R})$. More precisely:

Lemma 2.4 (Smoothness Lemma) *Let*

$$A : H^\infty(\mathbb{R}) \to H^\infty(\mathbb{R})$$

be a continuous linear operator. Given $q > r + 1/2$ and $r \geq 1$, suppose that A extends to a bounded operator from $H^q(\mathbb{R})$ to $H^{q-r}(\mathbb{R})$. Then:

$$\varphi \mapsto A_\varphi := R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q \to \mathcal{L}(H^q(\mathbb{R}), H^{q-r}(\mathbb{R}))$$

is smooth if and only if each operator A_n defined by (2.5), extends to a bounded $(n+1)$-linear operator in $\mathcal{L}^{n+1}(H^q(\mathbb{R}), H^{q-r}(\mathbb{R}))$.

According to [3, Prop. 6] the operator A_{n+1} can be written in terms of the multilinear commutators:

$$S_{n,P}(u_1, u_2, \dots, u_n) := [u_1, [u_2 \cdots [u_n, P] \cdots]],$$

for some operators P. Therefore the proof of Proposition 2.3 reduces to proving the following result:

Proposition 2.5 *Let $A = \sigma_A(X, D) \in \mathcal{E}^r$ with an x-compactly suported symbol σ_A, then each A_n extends to a bounded multi-linear operator:*

$$A_n \in \mathcal{L}^{n+1}(H^q(\mathbb{R}), H^{q-r}(\mathbb{R})).$$

Remark 2.6 *Of course, some of the propositions presented above still hold in a more general context, but for our goal we restricted to the class $\mathcal{E}^r$, even when some of the restrictions imposed are redundant.*

3 A Taylor type Estimate

In this section we prove the boundedness result that lies at the root of Proposition 2.3 and implies indirectly the aforementioned well-posedness result:

Proposition 3.1 *Let $P \in OpS^{r+n-1}_{1,0}$ with a hermitian symbol compactly supported in x. For $w \in \mathrm{C}^\infty_c(\mathbb{R}), u_1, \dots, u_n \in \mathrm{C}^\infty_c(\mathbb{R})$, we have:*

$$\|S_{n,P}(u_1, \dots, u_n)w\|_{H^{q-r}} \lesssim \|u_1\|_{H^q} \cdots \|u_n\|_{H^q} \|w\|_{H^{q-1}},$$

where $q > r + 1/2$, $r \geq 1$, and:

$$S_{n,P}(u_1, u_2, \dots, u_n) := [u_1, [u_2 \cdots [u_n, P] \cdots]].$$

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Since $C_c^\infty(\mathbb{R})$ is dense in $H^q(\mathbb{R})$ the above estimate will furnish the boundedness result of A_n, mentioned in Proposition 2.5. In the case $n = 1$, for a single commutator $S_{1,P}(u_1) := [u_1, P]$, the Sobolev boundedness has been proved by M. Taylor in [10], for a general pseudo-differential operator of class $OpS^r_{1,0}$. As far as we know the result can not be extended straightforward to multilinear commutators. Boundedness results like this one use para-differential calculus and seem to be a difficult task, in the case of a general pseudo-differential operator. The restriction to the class of pseudo-differential operators with x-compactly supported symbols, paved us the way to a relative simple proof. We present below the main arguments. The following lemma is standard:

Lemma 3.2 *For a smooth function u and a symbol p from the class $S^r_{1,0}$ the commutator $[u, P]$ has the symbol:*

$$\sigma_{[u,P]}(x,\xi) = \int_{\mathbb{R}} e^{2\pi i\cdot x\cdot\eta}\hat{u}(\eta)[p(x,\xi) - p(x,\xi+\eta)]d\eta,$$

where $\hat{u}$ is the Fourier transform.

Using this lemma one gets a formula for the symbol of $S_{n,P}(u_1, \dots u_n)$:

Proposition 3.3 *Given an operator $P \in OpS^{r+n-1}$, $r \geq 1$, the following formula holds:*

$$\sigma_{S_{n,P}(u_1,\dots u_n)}(x,\xi) =$$

$$\int_{\mathbb{R}} e^{2\pi i x\cdot\eta}\int_{\xi_1+\dots+\xi_n=\eta} \hat{u}_1(\xi_1)\hat{u}_2(\xi_2)\dots\hat{u}_n(\xi_n)\cdot p_n(x,\xi,\xi_1,\dots\xi_n)d\mu d\eta,$$

where μ is the Lebesgue measure on the subspace $\xi_1 + \dots + \xi_n = \eta$ of $\mathbb{R}^{n+1}$ and:

$$p_n(x,\xi,\xi_1,\xi_2,\dots\xi_n) = \sum_{\substack{J\subseteq I_n\\ |J|=k}} (-1)^k p(x,\xi+\sum_{j\in J}\xi_j),$$

for p the symbol of P, with $u_1, \dots u_n \in C_0^\infty(\mathbb{R},\mathbb{C})$ and $I_n := \{1, 2, \dots, n\}$.

In order to prove the above proposition and to study the boundedness of the multilinear operator S_n a few estimates, on p_n and on its Fourier coefficient $\widehat{p_n}$, are necessary:

Lemma 3.4 *Let $p \in S^{r+n-1}$, $r \geq 1$, an x-compactly supported symbol. The estimate holds:*

$$\|p_n(x,\xi,\xi_1,\dots\xi_n)\| \leq C_{n,r}\langle\xi\rangle^{r-1}\langle\xi_1\rangle^r\dots\langle\xi_n\rangle^r, \tag{3.1}$$

where $C_{n,r}$ depends only on $n \in \mathbb{N}$ and r. Moreover we have the intermediary estimates:

$$\|p_s(x,\xi,\xi_1,\dots\xi_s)\| \leq C_{s,r}\langle\xi\rangle^{r-1+(n-s)}\langle\xi_1\rangle^{r+(n-s)}\dots\langle\xi_s\rangle^{r+(n-s)}, \tag{3.2}$$

for every $s = \overline{1,n}$. Finally for every $t \in \mathbb{N}$ there is a constant $C_n > 0$ such that:

$$|\widehat{p_n}(\eta,\xi,\xi_1\dots\xi_n)| \leq C_n\langle\eta\rangle^{-t}\langle\xi\rangle^{r-1}\langle\xi_1\rangle^r\dots\langle\xi_n\rangle^r, \tag{3.3}$$

and C_n is independent on $\eta, \xi, \xi_1, \dots \xi_n \in \mathbb{R}$.

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Proof 3.5 *In order to obtain the first two estimates we use a similar technique to the one used to Lemma A.6 in [5], because:*

$$\|\partial_\xi^\beta p(x,\xi)\| \leq C_n \langle \xi \rangle^{r-1},$$

for $|\beta| = n$.

The operator corresponding to the symbol $\langle \xi \rangle^2 := 1 + \xi^2$ *is* $I - \frac{1}{(2\pi)^2} D_x^2$ *and:*

$$\left(I - \frac{1}{(2\pi)^2} D_x^2 \right)^q (e^{-2\pi i \cdot x \cdot \xi}) = \langle \xi \rangle^{2q} \cdot e^{-2\pi i \cdot x \cdot \xi},$$

for every $q \in \mathbb{N}^*$. *Thus:*

$$\begin{aligned} |\widehat{p_n}(\eta, \xi, \xi_1 \dots \xi_n)| &= \left| \int_{\mathbb{R}} e^{-2\pi i \eta \cdot x} p_n(x, \xi, \xi_1, \dots \xi_n) dx \right| \\ &= \langle \eta \rangle^{-2q} \left| \int_{\mathbb{R}} \langle \eta \rangle^{2q} e^{-2\pi i \eta \cdot x} p_n(x, \xi, \xi_1, \dots \xi_n) dx \right| \\ &= \langle \eta \rangle^{-2q} \left| \int_{\mathbb{R}} \left(I - \frac{1}{(2\pi)^2} D_x^2 \right)^q (e^{-2\pi i \eta \cdot x}) p_n(x, \xi, \xi_1, \dots \xi_n) dx \right|. \end{aligned}$$

Since p_n *is* x*-compactly supported, an integration by parts leads to:*

$$\begin{aligned} |\widehat{p_n}(\eta, \xi, \xi_1 \dots \xi_n)| &= \langle \eta \rangle^{-2q} \left| \int_{\mathbb{R}} e^{-2\pi i \eta \cdot x} \left(I - \frac{1}{(2\pi)^2} D_x^2 \right)^q p_n(x, \xi, \xi_1, \dots \xi_n) dx \right| \\ &\leq \langle \eta \rangle^{-2q} \int_{\mathbb{R}} \left| \left(I - \frac{1}{(2\pi)^2} D_x^2 \right)^q p_n(x, \xi, \xi_1, \dots \xi_n) \right| dx \\ &\leq C_n \langle \eta \rangle^{-2q} \langle \xi \rangle^{r-1} \langle \xi_1 \rangle^r \dots \langle \xi_n \rangle^r, \end{aligned}$$

because the estimate (3.1) *on* p_n *is not affected by a derivative in* x. *For the estimate corresponding to* $t = 2q+1, q \in \mathbb{N}$, *one has to apply a square root to the product obtained from the estimates for* $t = 2q$ *and* $t = 2q + 2$.

Proof of Proposition 3.3: For the pseudo-differential operator $[u_2, [u_1, P]]$ applying Lemma 3.2:

$$\begin{aligned} \sigma_{[u_2,[u_1,P]]}(x,\xi) &= \int_{\mathbb{R}} e^{2\pi i x \cdot \xi_2} \hat{u}_2(\xi_2) \cdot \left[\sigma_{[u_1,P]}(x,\xi) - \sigma_{[u_1,P]}(x, \xi + \xi_2) \right] d\xi_2 \\ &= \int_{\mathbb{R}} e^{2\pi i x \cdot \xi_2} \hat{u}_2(\xi_2) \cdot \int_{\mathbb{R}} e^{2\pi i x \cdot \xi_1} \hat{u}_1(\xi_1) \cdot p_2(x, \xi, \xi_1, \xi_2) d\xi_1 d\xi_2 \end{aligned}$$

for:

$$p_2(x, \xi, \xi_1, \xi_2) := \sum_{\substack{J \subseteq I_2 \\ |J| = k}} (-1)^k p(x, \xi + \sum_{j \in J} \xi_j).$$

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We can use the co-area formula in the last expression to obtain:

$$\sigma_{[u_2,[u_1,P]]}(x,\xi)=\int_{\mathbb{R}}e^{2\pi ix\cdot\eta}\cdot\int_{\xi_1+\xi_2=\eta}\hat{u}_1(\xi_1)\hat{u}_2(\xi_2)\cdot p_2(x,\xi,\xi_1,\xi_2)d\mu d\eta$$

The integrand belongs to $\mathcal{L}^1$ by Hölder's inequality, using similar arguments with those of Corollary A.7 in [5]:

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\xi_1+\xi_2=\eta}\left|e^{2\pi ix\cdot\eta}\hat{u}_1(\xi_1)\hat{u}_2(\xi_2)\cdot p_2(x,\xi,\xi_1,\xi_2)\right|d\mu d\eta\\&\leq\int_{\mathbb{R}}\int_{\xi_1+\xi_2=\eta}|\hat{u}_1(\xi_1)|\cdot|\hat{u}_2(\xi_2)|\cdot|p_2(x,\xi,\xi_1,\xi_2)|d\mu d\eta\\&\leq\left(\int_{\mathbb{R}}\langle\tau\rangle^{-2q}d\tau\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\langle\tau\rangle^{2q}\left(\int_{\xi_1+\xi_2=\eta}|\hat{u}_1(\xi_1)|\cdot|\hat{u}_2(\xi_2)|\cdot|p_2(x,\xi,\xi_1,\xi_2)|d\mu\right)^2d\tau\right)^{\frac{1}{2}}\end{aligned}$$

which is estimated by $\langle\xi\rangle^{r-1+(n-2)}|u_1|_{H^{q+r+(n-2)}}|u_2|_{H^{q+r+(n-2)}}$, for $q>\frac{d}{2}$.

Applied successively, the same idea leads to the symbol formula of $S_{n,P}(u_1,\dots u_n)$.

We are now ready to prove the main result of this section:

Proof of Proposition 3.1: Our previous arguments let us write:

$$\begin{aligned}&S_{n,P}(u_1,\dots,u_n)(w)\\&=\int_{\mathbb{R}}\int_{\mathbb{R}}e^{2\pi ix\cdot(\xi_0+\eta)}\int_{\xi_1+\dots+\xi_n=\eta}\hat{w}(\xi_0)\hat{u}_1(\xi_1)\dots\hat{u}_n(\xi_n)p_n(x,\xi_0,\dots,\xi_n)d\mu d\eta d\xi_0\\&=\int_{\mathbb{R}}e^{2\pi ix\cdot\eta}\int_{\xi_0+\dots+\xi_n=\eta}\hat{w}(\xi_0)\hat{u}_1(\xi_1)\dots\hat{u}_n(\xi_n)p_n(x,\xi_0,\dots,\xi_n)d\overline{\mu}d\eta\,,\end{aligned}$$

where $\overline{\mu}$ is the Lebesgue measure on the subspace $\xi_0+\xi_1+\ldots+\xi_n=\eta$ of $\mathbb{R}^{n+2}$. Expressing $p_n(x,\xi_0,\dots,\xi_n)$ via its Fourier transform yields:

$$\begin{aligned}&S_{n,P}(u_1,\dots,u_n)(w)\\&=\int_{\mathbb{R}}\int_{\mathbb{R}}e^{2\pi ix\cdot(\eta+\zeta)}\int_{\xi_0+\dots+\xi_n=\eta}\hat{w}(\xi_0)\dots\hat{u}_n(\xi_n)\hat{p}_n(\zeta,\xi_0,\dots,\xi_n)d\overline{\mu}d\eta d\zeta\\&=\int_{\mathbb{R}}e^{2\pi ix\cdot\zeta}\int_{\mathbb{R}}\int_{\xi_0+\dots+\xi_n=\eta}\hat{w}(\xi_0)\dots\hat{u}_n(\xi_n)\hat{p}_n(\zeta-\eta,\xi_0,\dots,\xi_n)d\overline{\mu}d\eta d\zeta\,.\end{aligned}$$

Hence, denoting also by $\mathfrak{F}$ the Fourier transform, one gets:

$$\begin{aligned}&\mathfrak{F}\left(S_{n,P}(u_1,\dots,u_n)(w)\right)(\zeta)\\&=\int_{\mathbb{R}}\int_{\xi_0+\dots+\xi_n=\eta}\hat{w}(\xi_0)\dots\hat{u}_n(\xi_n)\hat{p}_n(\zeta-\eta,\xi_0,\dots,\xi_n)d\overline{\mu}d\eta\,.\end{aligned}$$

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Next we estimate via Peetre's inequality $\langle\zeta\rangle^{q-r}\lesssim\langle\zeta-\eta\rangle^{q-r}\langle\eta\rangle^{q-r}$ and (3.3):

$$\begin{aligned}&\left|\langle\zeta\rangle^{s-r}\mathfrak{F}\left(S_{n,P}(u_1,\ldots,u_n)(w)\right)(\zeta)\right|\\&\lesssim\int_{\mathbb{R}}\langle\zeta-\eta\rangle^{q-r-t}\langle\eta\rangle^{s-r}\int_{\xi_0+\cdots+\xi_n=\eta}\langle\xi_0\rangle^{r-1}|\hat{w}(\xi_0)|\prod_{i=1}^{n}\langle\xi_i\rangle^{r}|\hat{u}_i(\xi_i)|d\overline{\mu}d\eta\,,\end{aligned}$$

for some $t\in\mathbb{N}$. For this is a convolution, we can use Young's inequality to estimate:

$$\begin{aligned}&\left\|\langle\zeta\rangle^{q-r}\mathfrak{F}\left(S_{n,P}(u_1,\ldots,u_n)(w)\right)(\zeta)\right\|_{L^2}^2\\\lesssim&\left(\int_{\mathbb{R}}\langle\zeta-\eta\rangle^{q-r-t}d\eta\right)^2\cdot\int_{\mathbb{R}}\langle\eta\rangle^{2(q-r)}\left(\int_{\xi_0+\cdots+\xi_n=\eta}\langle\xi_0\rangle^{r-1}|\hat{w}(\xi_0)|\prod_{i=1}^{n}\langle\xi_i\rangle^{r}|\hat{u}_i(\xi_i)|d\overline{\mu}\right)^2d\eta\,.\end{aligned}$$

and the first integral $\int_{\mathbb{R}}\langle\zeta-\eta\rangle^{q-r-t}d\eta$ is bounded since t can be chosen big enough. Let Λ be the operator, such that $\mathfrak{F}\left(\Lambda^r u\right)(\xi)=\langle\xi\rangle^r\mathfrak{F}(u)(\xi)$, which is actually the Fourier multiplier with symbol $\langle\xi\rangle^r$. Let us consider the functions defined by $\tilde{w}(x)=\mathfrak{F}^{-1}(|\hat{w}(\xi_0)|)(x)$ and $\tilde{u}_i(x)=\mathfrak{F}^{-1}(|\hat{u}_i(\xi_i)|)(x)$. First of all, as a consequence of Plancherel's theorem the Sobolev norms of $\tilde{w}$ and w will coincide.

After applying the identity:

$$\mathfrak{F}(f_0\cdot f_1\cdot\ldots f_n)(\eta)=\int_{\xi_0+\cdots+\xi_n=\eta}\hat{f}_0(\xi_0)\cdot\hat{f}_1(\xi_1)\cdot\ldots\cdot\hat{f}_n(\xi_n)d\overline{\mu}$$

to $f_0=\Lambda^{r-1}\tilde{w}$ and $f_i=\Lambda^r\tilde{u}_i,\ i=\overline{1,n}$ we obtain:

$$\int_{\xi_0+\cdots+\xi_n=\eta}\langle\xi_0\rangle^{r-1}|\hat{w}(\xi_0)|\prod_{i=1}^{n}\langle\xi_i\rangle^{r}|\hat{u}_i(\xi_i)|d\overline{\mu}\qquad=\qquad\mathfrak{F}\left(\Lambda^{r-1}\tilde{w}\cdot\prod_{i=1}^{n}\Lambda^r\tilde{u}_i\right)(\eta)\,,$$

Thus:

$$\left\|\langle\zeta\rangle^{q-r}\mathfrak{F}\left(S_{n,P}(u_1,\ldots,u_n)(w)\right)(\zeta)\right\|_{L^2}\lesssim\left\|\Lambda^{r-1}\tilde{w}\cdot\prod_{i=1}^{n}\Lambda^r\tilde{u}_i\right\|_{H^{q-r}}$$

and:

$$\left\|S_{n,P}(u_1,\ldots,u_n)(w)\right\|_{H^{q-r}}\lesssim\|w\|_{H^{q-1}}\cdot\prod_{i=1}^{n}\|u_i\|_{H^q},$$

since for $q-r>\frac{1}{2}$ the space $H^{q-r}(\mathbb{R})$ is an algebra.

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4 Concluding Remarks

The well-posedness result presented in Corollary 2.2 is also true for $\mathbb{R}^d$ when the Euler-Poincaré equation becomes the EPDiff equation:

$$m_t + \nabla_u m + (\nabla u)^t\, m + (\operatorname{div} u)m = 0, \quad m := Au\,, \tag{4.1}$$

with important applications in shape analysis or computational anatomy, see [12]. In order to keep our presentation reasonably short and to avoid some tedious computations we discussed only the case $d = 1$. It is still not very clear if one can drop the restriction $r \geq 1$, for the order of the inertia operator A, since the same restriction occurs with a Nash-Moser approach in the case of the torus S^1.

References

[1] V. I. Arnold. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier (Grenoble)*, 16(fasc. 1):319–361, 1966.

[2] M. Bauer and J. Escher and B. Kolev Local and global well-posedness of the fractional order EPDiff equation on R^d. *Journal of Differential Equations*, 258: 2010–2053, 2015.

[3] E.C. Cismas, Euler-Poincaré-Arnold equations on semi-direct products II

Discrete and Continuous Dynamical Systems- Series A, **36** (2016), 5993-6022

[4] D. G. Ebin and J. E. Marsden. Groups of diffeomorphisms and the notion of an incompressible fluid. *Ann. of Math. (2)*, 92:102–163, 1970.

[5] J. Escher and B. Kolev. Right-invariant Sobolev metrics of fractional order on the diffeomorphisms group of the circle. *Journal of Geometric Mechanics.* Vol. 6, No. 3, 335–372, 2014.

[6] J. Escher and B. Kolev Geometrical methods for equations of hydrodynamical type. *J. Nonlinear Math.Phys.*, Vol 19, Suppl 1, 2012.

[7] R. S. Hamilton. The inverse function theorem of Nash and Moser. *Bull. Amer. Math. Soc. (N.S.)*, 7(1):65–222, 1982.

[8] A. Himonas, C Kenig. Non-uniform dependence on initial data for the CH equation on the line. *Differential and Integral Equations* 22, No. 3-4, 201-224, 2009.

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[9] S. Lang, *Fundamentals of Differential Geometry*, Graduate Texts in Mathematics,, **191**, Springer-Verlag, New York, 1999.

[10] M. E. Taylor Commutator estimates. *Proceedings of the American Mathematical Society* Vol. 131, No. 5, 1501–1507, 2002.

[11] M. E. Taylor. Pseudodifferential Operators and Nonlinear PDE. *Springer*, 2013.

[12] L. Younes, F. Arrate, M. Miller. Evolution equations in computational anatomy. *NeuroImage* , 45, S40–S50, 2009.

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