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PLSM FOR BAGLEY TORVIK EQUATION MODELING THE DEFORMATION RESISTANCE CHARACTERISTICS OF THE POLYMER CONCRETE

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Abstract

In this paper we study the Bagley Torvik fractional differential equation which models the deformation resistance characteristics of the polymer concrete. An approximate analytical solution for this equation is obtaining using the Polynomial Least Squares Method (*PLSM*). [1]

Keywords and phrases: *Fractional differential equations, Polynomial Least Square Method(PLSM).*

1 Introduction

Fractional differential equations are widely used in many branches of science such as mathematics, physics, chemistry, biology, and this is the reason why in recent years these equations have been investigated by many authors. In this paper oscillation processes in a viscoelastic medium are described using the fractional Bagley Torvik equation. Using the Polynomial Least Squares Method we obtain an approximate analytical solution of the Bagley Torvik equation, which, compared to the numerical and experimental results for polymer concrete samples presented in ([1]), demonstrates the accuracy of the method. The equation (1) was introduced by Bagley and Torvik in 1983 ([4],[5]) in order to modeling the damping properties of various elastic-plastic materials (polymers, glasses, etc.). In the years that followed, various methods were used to solve numerically or analytically this equation, among which we mention: Homotopy perturbation

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method ([10]), Discrete spline methods ([6], [8]), Collocation method ([7]), Hybrid functions approximation ([9]).

In engineering, an important role in the development of new materials used in industry, medicine, construction etc is played by polymers. A polymer is a chemical compound with large molecules made of many smaller molecules of the same kind. Some polymers exist naturally and others are produced in laboratories and factories.

In one of the recent paper Temirkhan Aleroev et all ([1]), the Bagley-Torvik equation, presented as:

$$y"(x) + k \cdot D_x^{\alpha} y(x) + \lambda \cdot y(x) = 0, \tag{1}$$

with $1 < \alpha < 2$, and

$$y(0) = 0, y'(0) = 1, \tag{2}$$

was used in modeling the change in the deformation- strength characteristics of polymer concrete when subjected to loadings. Polymer concrete is represented as a set of granules of mineral extender in an elastic-plastic medium. In this case, the motion of the granule is described by the equation (1), where k is the viscosity modulus of the resin, λ is the rigidity modulus of the resin and α is the elastic-plastic parameter of the medium.

In this paper we obtain an analytical approximate solution for the equation (1) where $D_x^{\alpha} y(x)$ denotes Caputo's fractional derivative of order α:

$$D_x^{\alpha} y(x) = \frac{1}{\Gamma(2-\alpha)} \cdot \int_0^x (x-\zeta)^{-(\alpha+1)} \cdot y^{(2)}(\zeta) d\zeta, \qquad 1 < \alpha < 2. \tag{3}$$

In the next section we will introduce the *Polynomial Least Square Method* (*PLSM*) ([2], [3]) which allows us to determine analytical approximate polynomial solutions for fractional ordinary differential equations and in the third section we will compare our approximate solutions with the numerical data presented by Temirkhan Aleroev et all in [1]).

2 The Polynomial Least Squares Method

We denote by $\tilde{y}(x)$ an approximate solution of equation (1). The error obtained by replacing the exact solution $y(x)$ with the approximation $\tilde{y}(x)$ is given by the remainder:

$$\mathcal{R}(\tilde{y}(x)) = \tilde{y}"(x) + k \cdot D_x^{\alpha} \tilde{y}(x) + \lambda \cdot \tilde{y}(x). \tag{4}$$

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For $\epsilon \in \mathbb{R}_+$, we will compute approximate polynomial solutions $\tilde{y}(x)$ of the problem $(1,2)$ on the interval $[0,2]$.

Definition 2.1. *We call an* ϵ-approximate polynomial solution *of the problem* $(1,2)$ *an approximate polynomial solution* $\tilde{y}(x)$ *satisfying the relations*

$$|\mathcal{R}(\tilde{y})| < \epsilon \tag{5}$$

with

$$\tilde{y}(0) = 0. \tag{6}$$

We call a *weak* ϵ*-approximate polynomial solution* of the problem $(1,2)$ an approximate polynomial solution $\tilde{y}(x)$ satisfying the relation:

$$\int_0^2 |\mathcal{R}(\tilde{y})| dx \leq \epsilon \tag{7}$$

together with the condition (6).

Definition 2.2. *Let* $P_m(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_m x^m,\ c_i \in \mathbb{R},\ i = \overline{0,m}$ *be a sequence of polynomials satisfying the condition:*

$$P_m(0) = 0.$$

We call the sequence of polynomials $P_m(x)$ *convergent to the solution of the problem* $(1,2)$ *if* $\lim_{m\to\infty} \mathcal{R}(P_m(x)) = 0$.

We observe that from the hypothesis of the initial problems $(1,2)$ it follows that there exists a sequence of polynomials $P_m(x)$ which converges to the solution of the problem, according to the Weierstrass Theorem on Polynomial Approximation.

We will compute a weak ϵ - approximate polynomial solution, in the sense of the Definition 2.1, of the type:

$$\tilde{y}(x) = \sum_{k=0}^{m} d_k x^k, m > 0 \tag{8}$$

where $d_0, d_1, \cdots, d_m$ are constants which are calculated using the following steps:

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- By substituting the approximate solution (8) in the equation (1) we obtain the expression:

$$\mathcal{R}(\tilde{y}) = \tilde{y}"(x) + k \cdot D_x^{\alpha}\tilde{y}(x) + \lambda \cdot \tilde{y}(x). \tag{9}$$

If we could find $d_0, d_1, \cdots, d_m$ such $\mathcal{R}(\tilde{y}) = 0,\ \tilde{y}(0) = 0$, then by substituting $d_0, d_1, \cdots, d_m$ in (7) we obtain the solutions of equation (1).

- We attach to the problem (1,2) the following functional:

$$\mathcal{J}(d_1, d_2, d_3, \cdots, d_m) = \int_0^2 \mathcal{R}^2(\tilde{y})dx \tag{10}$$

where d_0 is computed as functions of $d_1, d_2, d_3, \cdots, d_m$ using the initial condition (6).

- We compute the values $d_1^0, d_2^0, d_3^0, \cdots, d_m^0$ as the values which give the minimum of the functional $\mathcal{J}$, and the values of d_0 is function of $d_1^0, d_2^0, d_3^0, \cdots, d_m^0$ using the initial condition.

- With constants $d_1^0, d_2^0, d_3^0, \cdots, d_m^0$ previously determined we consider the polynomial:

$$T_m(x) = \sum_{k=0}^{m} d_k^0 x^k. \tag{11}$$

Theorem 2.1. *The sequence of polynomials $T_m(x)$ from (11) satisfies the property:*

$$\lim_{m\to\infty} \int_0^2 \mathcal{R}^2(T_m(x))dx = 0. \tag{12}$$

Moreover, $\forall \epsilon > 0,\ \exists m_o \in \mathbb{N},\ m > m_0$ it follows that $T_m(x)$ is a weak ϵ-approximate polynomial solution *of the problem* $(1, 2)$.

Proof. Based on the way the polynomials $T_m(x)$ are computed and taking into account the relations (9)-(12), the following inequalities are satisfied:

$$0 \le \int_0^2 \mathcal{R}^2(T_m(x))dx \le \int_0^2 \mathcal{R}^2(P_m(x))dx,\ \forall m \in \mathbb{N},$$

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where $P_m(x)$ is the sequence of polynomials introduced in Definition 2.2.

It follows that:

$$0 \leq \lim_{m\to\infty} \int_0^2 \mathcal{R}^2(T_m(x))dx \leq \lim_{m\to\infty} \int_0^2 \mathcal{R}^2(P_m(x))dx = 0.$$

and:

$$\lim_{m\to\infty} \int_0^2 \mathcal{R}^2(T_m(x))dx = 0.$$

We obtain that $\forall \epsilon > 0,\ \exists m_o \in \mathbb{N},\ m > m_0$ it follows that $T_m(x)$ is a *weak ϵ-approximate polynomial solution* of the problem $(1, 2)$. □

In order to find ϵ-approximate polynomial solutions of the problem $(1, 2)$ by using the Polynomial Least Squares Method we will first determine a weak approximate polynomial solutions, $\tilde{y}(x)$. If $|\mathcal{R}(\tilde{y}(x))| < \epsilon$ then $\tilde{y}(x)$ is also an ϵ approximate polynomial solution of the problem.

3 Application

In the equation (1) the remaining parameters are $k = 1.8$ and $\lambda = 93$. These parameter values were obtained during experiments on samples of polymer concrete and the results was presented by Alerov in ([2]).

We consider the following Bagley-Torvik fractional differential equation ([2]):

$$y"(x) + 1.8 \cdot D_x^{\alpha} y(x) + 93 \cdot y(x) = 0 \tag{13}$$

with $x \in [0, 2]$, $y(0) = 0$ and $y'(0) = 1$.

Using the Polynomial Least Squares Method (PLSM) we follow the steps outlined in the previous section:

- We compute a solution of the type:
  $\tilde{y}(x) = d_0 + d_1 \cdot x^1 + d_2 \cdot x^2 + d_3 \cdot x^3 + d_4 \cdot x^4 + d_5 \cdot x^5 + d_6 \cdot x^6 + d_7 \cdot x^7 +$
  $+ d_8 \cdot x^8 + d_9 \cdot x^9 + d_{10} \cdot x^{10} + d_{11} \cdot x^{11} + d_{12} \cdot x^{12}$
  and from initial conditions: $\tilde{y}(0) = 0$ and $\tilde{y}'(0) = 1$ we obtain: $d_0 = 0.$ and $d1 = 1.$

- The approximate solution becomes:
  $\tilde{y}(x) = d_1 \cdot x^1 + d_2 \cdot x^2 + d_3 \cdot x^3 + d_4 \cdot x^4 + d_5 \cdot x^5 + d_6 \cdot x^6 + d_7 \cdot x^7 + d_8 \cdot x^8 +$
  $+ d_9 \cdot x^9 + d_{10} \cdot x^{10} + d_{11} \cdot x^{11} + d_{12} \cdot x^{12}.$

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- The corresponding remainder is $\mathcal{R}(\tilde{y}(x))$ (whose expression is too long to be introduced here). Next we compute:

$$\mathcal{J}(d_2, d_3, \cdots, d_9) = \int_0^2 \mathcal{R}^2(\tilde{y}(x))dx$$

and minimize it obtaining the values:
$d_2 \to -0.0858228, d_3 \to -11.1994, d_4 \to -3.146, d_5 \to 100.95, d_6 \to -206.366,$
$d_7 \to 184.913, d_8 \to -73.859, d_9 \to 0.148445, d_{10} \to 11.0816, d_{11} \to -3.81496$
and $d_{12} \to 0.425751$

- The approximate analytical solution of the problem (13) using (*PLSM*) is:
$\tilde{y}(x) = 0.425751x^{12} - 3.81496x^{11} + 11.0816x^{10} + 0.148445x^9 - 73.859x^8 +$
$184.913x^7 - 206.366x^6 + 100.95x^5 - 3.146x^4 - 11.1994x^3 - 0.0858228x^2 + x.$

Alerov et all in ([1]) presented some experimental results and also a numerical solution for the problem (13).

Table 1 present the comparison between experimental results corresponding to the numerical solution proposed by Alerov in ([1]) and our solution obtained using (*PLSM*).

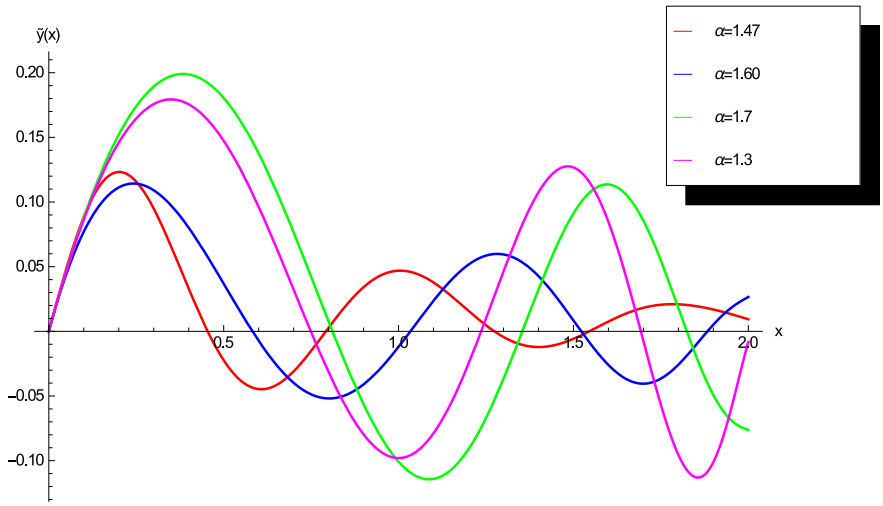
Table 1: Numerical results

x	Experimental results	$PLSM$	$\lvert Experimental - PLSM\rvert$
0.25	5×10^{-2}	1.171×10^{-1}	6.716×10^{-2}
0.50	-4×10^{-2}	-2.308×10^{-2}	1.691×10^{-2}
0.75	-1×10^{-2}	-1.559×10^{-2}	5.597×10^{-3}
1.00	2×10^{-2}	4.786×10^{-2}	2.786×10^{-2}
1.25	-1×10^{-2}	3.868×10^{-3}	1.386×10^{-2}
1.50	-1×10^{-2}	$-5..273 \times 10^{-2}$	4.726×10^{-3}

Figure 1 shows the graph of the approximate analytical solution $\tilde{y}(x)$, for different values of α between 1 and 2 (for each approximation a 12 degree polynomial is used).

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Figure 1: Graphs of solutions when $1 < \alpha < 2$



4 Conclusions

In the present paper using (PLSM) an analytical solution was obtained for the fractional differential equation which models the deformation-strength characteristics of polymer concrete. The computations performed show that the restults obtained by using (*PLSM*) are in good agreement with the experimental or numerical results obtained by Alerov et all in ([2]). The results were obtained in an easy manner, using minimal time resources, the calculations being made in Wolfram Mathematica.

Additionally, using the Polynomial Least Squares Method one obtains the analytical solution of the problem, not only numerical solutions, fact which demonstrates the usefulness of the (PLSM).

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