SCIENTIFIC BULLETIN
OF
THE POLITEHNICA UNIVERSITY OF TIMISOARA

Vol. 64(78), Issue 1 MATHEMATICS–PHISYCS 2019

APPROXIMATE SOLUTIONS BY THE LEAST SQUARES DIFFERENTIAL QUADRATURE METHOD FOR NONLINEAR HEAT TRANSFER PROBLEMS

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Abstract

In the present paper we employ a recently introduced approximation method, namely the Least Squares Differential Quadrature Method (LSDQM), in order to compute analytical approximate polynomial solutions for several nonlinear heat transfer problems. [1]

Keywords and phrases: *Nonlinear heat transfer problems, Least Squares Differential Quadrature Method (LSDQM).*

1 Introduction

The heat transfer phenomena are mainly nonlinear and thus they are best modeled by using nonlinear equations. The vast majority of these equations can not be solved analytically using traditional methods and, when the numerical solution of the nonlinear problem is not sufficient, an approximate analytical solution must be computed. In recent years many methods have been developed to compute approximate solutions for nonlinear equations. Among these methods we mention: the Homotopy perturbation method (HPM)([1],[2],[3]), the Homotopy analysis method (HAM)([4],[5]), the Optimal Homotopy Asymptotic Method (OHAM)([6]), the Generalized approximation method (GAM)([7]), the Squared remainder minimization method (SRMM)([8]).
In this paper we compute approximate analytical solutions for some well-known nonlinear heat transfer problems ([15],[16],[17]) modeled by using nonlinear ordinary differential equations.

[1]MSC(2010): 34K28, 45L05

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In the next section we will describe the Least Squares Differential Quadrature Method (LSDQM), which allows us to determine analytical approximate polynomial solutions for the above-mentioned type of problems, and in the third section we will compare approximate solutions obtained by using LSDQM with previous approximate solutions presented in ([6],[7],[8],[9]). The computations show that by using LSDQM we obtain approximations with an error smaller than the errors obtained by using other methods.

2 The Least Squares Differential Quadrature Method (LSDQM)

We consider a problem consisting of a nonlinear differential equation of order n:

$$u^{(n)}(t) = F(u^{(n-1)}(t), u^{(n-2)}(t), \cdots, u^{(1)}(t), u(t), t) \tag{1}$$

where F is a continuous function, $t \in [a, b]$ and the boundary conditions:

$$u^{(n-1)}(a) + u^{(n-2)}(a) + \cdots + u^{(1)}(a) + u(a) = \mu_1, \tag{2}$$

$$u^{(n-1)}(b) + u^{(n-2)}(b) + \cdots + u^{(1)}(b) + u(b) = \mu_2. \tag{3}$$

We will consider a numerical meshing of the interval $I = [a, b]$ by means of a partition Δ_M consisting of $M+1$ equidistant points: $a = t_0 < t_1 < t_2 < \cdots < t_{M-1} < t_M = b$. To the equation (1) we attach the following operator:

$$D(u(t)) = u^{(n)}(t) - F(u^{(n-1)}(t), u^{(n-2)}(t), \cdots, u^{(1)}(t), u(t), t). \tag{4}$$

We denote by $\tilde{u}(t)$ an approximate solution of the equation (1). By replacing in D the exact solution $u(t)$ with this approximate solution we obtain the *reminder*:

$$\mathcal{R}(t, \tilde{u}(t)) = D(\tilde{u}(t)), \qquad t \in [a, b]. \tag{5}$$

Definition 1. *We call an* ϵ***-approximate*** *solution of the problem (1 - 3) related to the partition* Δ_M *an approximate polynomial solution which satisfies the following relations:*

$$\mathcal{R}(t_i, \tilde{u}(t_i)) < \epsilon, \qquad i = \overline{0, M}, \tag{6}$$

$$\tilde{u}^{(n-1)}(a) + \tilde{u}^{(n-2)}(a) + \cdots + \tilde{u}^{(1)}(a) + \tilde{u}(a) = \mu_1, \tag{7}$$

$$\tilde{u}^{(n-1)}(b) + \tilde{u}^{(n-2)}(b) + \cdots + \tilde{u}^{(1)}(b) + \tilde{u}(b) = \mu_2. \tag{8}$$

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Definition 2. *We consider the sequence of polynomials:*

$$P_N(t) = \sum_{k=0}^{N} c_k t^k, \qquad c_k \in \mathbb{R}, \qquad k = \overline{0, N}. \tag{9}$$

We call the sequence of polynomials $P_N(t)$ ***convergent*** *to the solution of the problem (1 - 3) if:*

$$\lim_{N\to\infty} D(P_N(t)) = 0. \tag{10}$$

We will compute ϵ - approximate polynomial solutions of the type:

$$T_N(t) = \sum_{k=0}^{N} \tilde{c_k} t^k, \tag{11}$$

with the boundary conditions:

$$T_N^{(n-1)}(a) + T_N^{(n-2)}(a) + \cdots + T_N^{(1)}(a) + T_N(a) = \mu_1, \tag{12}$$

$$T_N^{(n-1)}(b) + T_N^{(n-2)}(b) + \cdots + T_N^{(1)}(b) + T_N(b) = \mu_2. \tag{13}$$

The constants $\tilde{c_k}$ are calculated taking the following steps:

- From the boundary conditions we obtain $\tilde{c_0}$ and $\tilde{c_1}$ as functions of $\tilde{c_2}, \tilde{c_3} \cdots \tilde{c_N}$ and replace them in the expression of $T_N(t)$ (which from now on will be a function of $\tilde{c}_2, \tilde{c}_3, \cdots, \tilde{c}_N$ only).
- We attach to the problem (1 - 3) the functional:

$$\mathcal{J}(\tilde{c}_2, \tilde{c}_3, \cdots, \tilde{c}_N) = \sum_{i=0}^{M} \mathcal{R}^2(t_i, T_N(t_i)). \tag{14}$$

- By minimizing the functional (14) we obtain the coefficients $\tilde{c_2}, \tilde{c_3} \cdots \tilde{c_N}$.
- We replace the coefficients $\tilde{c_2}, \tilde{c_3} \cdots \tilde{c_N}$.in the expression (11) and denote by $T_N^0(t) = \sum\limits_{k=0}^{N} \tilde{c_k} t^k$, the analytical approximate polynomial solutions by LSDQM of the problem (1 - 3).

The following convergence theorem is satisfied:

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Theorem 1. *The sequence of polynomials $T_N^0(t)$ satisfies the relations:*

$$\lim_{N\to\infty} \mathcal{R}^2(t_i, T_N^0(t_i)) = 0, \qquad i = \overline{0, M}. \tag{15}$$

Proof. Let $u(t)$ be an exact solution of the problem (1 - 3), which means from hypothesis that there exist a sequence of polynomials $P_N(t)$ with $P_N(t) = \sum_{k=0}^{N} c_k t^k$, $c_k \in \mathbb{R}$, $k\overline{0, N}$ converging to $u(t)$: $\lim_{N\to\infty} P_N(t) = u(t)$, $\forall t \in I$.

We know that $\sum_{i=0}^{M} \mathcal{R}^2(t_i, T_N^0(t_i)) \leq \sum_{i=0}^{M} \mathcal{R}^2(t_i, P_N(t_i))$, hence $\lim_{N\to\infty} (\sum_{i=0}^{M} \mathcal{R}^2(t_i, T_N^0(t_i))) \leq \lim_{N\to\infty} (\sum_{i=0}^{M} \mathcal{R}^2(t_i, P_N(t_i)))$.

We conclude that $\lim_{N\to\infty} \mathcal{R}^2(t_i, T_N^0(t_i)) = 0, \qquad i = \overline{0, M}.$ □

3 Applications

Application 1

We consider a lumped system of combined convective–radiative heat transfers. The specific heat coefficient is a linear function of temperature ([8],[9],[10]):

$$c = c_a(1 + \gamma(T - T_a))$$

where γ is a constant and c_a is the specific heat at T_a. The cooling process of the system is:

$$\rho V c \frac{dT}{d\tau} + hA(T - T_a) + E\sigma A(T^4 - T_s^4) = 0, \qquad T(0) = T_i$$

Performing the changes of variables: $u = \frac{T}{T_i}$, $u_a = \frac{T_a}{T_i}$, $t = \frac{\tau(hA)}{\rho V c_a}$, $\epsilon_1 = \gamma T_i$, $\epsilon_2 = \frac{E\sigma T_i^3}{h}$, $u_s = \frac{T_s}{T_i}$ and $u_a = u_s = 0$ we obtain the following problem:

$$\frac{du}{dt}(1 + \epsilon_1 u) + u + \epsilon_2 u^4 = 0, \qquad u(0) = 1 \tag{16}$$

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Case 1: $\epsilon_1 = 1, \epsilon_2 = 1$

Using the steps outlined in the previous section, we computed the following approximate solutions by LSDQM of the problem (17) :
- third order polynomial: $\tilde{u}(t) = -0.229672t^3 + 0.646583t^2 - 0.968634t + 1$;
- 6th order polynomial: $\tilde{u}(t) = 0.141072t^6 - 0.580724t^5 + 1.0409t^4 - 1.13478t^3 + 0.980631t^2 - 0.99987t + 1$;
- 8th order polynomial: $\tilde{u}(t) = 0.104895t^8 - 0.536881t^7 + 1.21968t^6 - 1.6701t^5 + 1.62301t^4 - 1.29063t^3 + 0.9973t^2 - 0.999999t + 1$

We will compare our approximate solutions with previous solutions: HPM obtained by Ganji et all in ([9]), HAM obtained by Damiary et all in ([11]) and $SRMM$ obtained by Caruntu and Bota in ([8]). Since Eq. (17) does not have a known exact solution, we computed for each approximate solution the relative error as the difference (in absolute value) between the approximate solution and the numerical solution given by the Wolfram Mathematica software.

Table 1 presents the comparison between the solutions obtained by different methods.

Table 1: Comparison of HPM, HAM, SRMM and LSDQ for $\epsilon_1 = \epsilon_2 = 1$

t	HPM	HAM	$SRMM3^{rd}$ deg	$LSDQ3^{rd}$ deg	$LSDQ6^{th}$ deg	$LSDQ8^{th}$ deg
0.1	1.2092	1.866×10^{-2}	1.704×10^{-3}	5.389×10^{-4}	5.073×10^{-5}	4.497×10^{-6}
0.2	7.247×10^{-1}	5.821×10^{-3}	4.66×10^{-4}	1.403×10^{-3}	4.131×10^{-5}	2.084×10^{-6}
0.3	4.276×10^{-1}	6.186×10^{-3}	1.193×10^{-3}	3.378×10^{-3}	8.371×10^{-7}	2.229×10^{-6}
0.4	2.472×10^{-1}	1.709×10^{-2}	2.129×10^{-3}	4.302×10^{-3}	2.565×10^{-6}	3.119×10^{-6}
0.5	1.389×10^{-1}	2.68×10^{-2}	1.995×10^{-3}	3.898×10^{-3}	2.9513×10^{-6}	1.682×10^{-6}
0.6	7.495×10^{-2}	3.528×10^{-2}	9.44×10^{-4}	2.385×10^{-3}	4.653×10^{-5}	9.547×10^{-7}
0.7	3.818×10^{-2}	4.253×10^{-2}	5.48×10^{-4}	3.062×10^{-4}	2.287×10^{-5}	2.216×10^{-6}
0.8	1.78×10^{-2}	4.857×10^{-2}	1.78×10^{-3}	1.578×10^{-3}	1.071×10^{-5}	1.898×10^{-6}
0.9	7.303×10^{-2}	5.345×10^{-2}	1.93×10^{-3}	2.355×10^{-3}	2.980×10^{-6}	4.789×10^{-7}

Case 2: $\epsilon_1 = 1, \epsilon_2 = 0$

Using LSDQM we computed the following second order polynomial approximate solution of equation (17): $\tilde{u}(t) = 0.068967t^2 - 0.501672t + 1$.

As in Case 1, we compare our solution with previous solutions: HPM obtained by Ganji in ([12]), HAM obtained by Abbasbandy in ([13]), $OHAM$ by Marinca and Herisanu in ([6]) and $SRMM$ by Caruntu and Bota in ([8]).

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Table 2: Comparison of HPM, HAM, OHAM, SRMM and LSDQ for $\epsilon_1 = 1, \epsilon_2 = 0$

t	HPM	HAM	$OHAM$	$SRMM2^{nd}$ deg	$LSDQ2^{nd}$ deg
0.1	3.350×10^{-2}	3.672×10^{-5}	3.708×10^{-2}	1.561×10^{-3}	1.076×10^{-4}
0.2	4.345×10^{-2}	1.954×10^{-3}	5.366×10^{-2}	8.024×10^{-5}	1.167×10^{-4}
0.3	4.029×10^{-2}	4.091×10^{-3}	5.658×10^{-2}	1.138×10^{-3}	5.698×10^{-5}
0.4	3.071×10^{-2}	5.415×10^{-3}	5.113×10^{-2}	1.669×10^{-3}	4.325×10^{-5}
0.5	1.886×10^{-2}	5.541×10^{-3}	4.118×10^{-2}	1.730×10^{-3}	1.571×10^{-4}
0.6	7.170×10^{-3}	4.432×10^{-3}	2.942×10^{-2}	1.379×10^{-3}	2.600×10^{-4}
0.7	3.062×10^{-3}	2.243×10^{-3}	1.762×10^{-2}	6.736×10^{-4}	3.292×10^{-4}
0.8	1.125×10^{-2}	7.827×10^{-4}	6.855×10^{-3}	3.304×10^{-4}	3.449×10^{-4}
0.9	1.726×10^{-2}	4.372×10^{-3}	2.320×10^{-3}	1.575×10^{-3}	2.903×10^{-4}

Case 3: $\epsilon_1 = 0, \epsilon_2 = 1$

Using the LSDQ, we computed the following approximate solutions:
- 5^{th} order polynomial approximate solution of equation (17):

$$\tilde{u}(t) = -1.37953t^5 + 4.44564t^4 - 5.63484t^3 + 3.84205t^2 - 1.97814t + 1,$$

and - 7^{th} order polynomial approximate solution

$$\tilde{u}(t) = -1.87875t^7 + 8.44452t^6 - 15.8346t^5 + 16.2513t^4 - 10.2726t^3 + 4.58368t^2 - 1.99919t + 1.$$

In Table 3 we present the comparison between the solutions obtained by LSDQM and the solution obtained by Rajabi et all in ([14]) using HPM, Domairry et all in ([11]) using HAM and Caruntu and Bota in ([8]) using $SRMM$.

Application 2

We consider the process of one-dimensional conduction in a slab of thickness L, with the two faces maintained at uniform temperatures T_1 and T_2 with $T_1 > T_2$. The thermal conductivity k is a linear function of temperature ([7], [11], [14]):

$$k = k_2(1 + \mu(T - T_2))$$

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Table 3: Comparison of HPM, HAM, SRMM and LSDQ for $\epsilon_1 = 0, \epsilon_2 = 1$

t	HPM	HAM	$SRMM5^{th}$ deg	$LSDQ5^{nd}$ deg	$LSDQ7^{th}$ deg
0.1	2.242×10^{-3}	1.188×10^{-5}	6.081×10^{-5}	2.527×10^{-3}	8.101×10^{-4}
0.2	9.474×10^{-3}	8.720×10^{-4}	1.256×10^{-3}	3.511×10^{-3}	3.789×10^{-4}
0.3	1.799×10^{-2}	1.251×10^{-3}	3.122×10^{-3}	1.400×10^{-3}	5.560×10^{-5}
0.4	2.513×10^{-2}	1.543×10^{-2}	9.676×10^{-3}	7.426×10^{-3}	1.189×10^{-4}
0.5	3.004×10^{-2}	2.093×10^{-2}	1.124×10^{-3}	1.064×10^{-3}	3.735×10^{-4}
0.6	3.270×10^{-2}	3.042×10^{-2}	1.784×10^{-4}	2.875×10^{-3}	3.066×10^{-4}
0.7	3.370×10^{-2}	4.351×10^{-2}	8.140×10^{-4}	1.831×10^{-4}	7.871×10^{-5}
0.8	3.350×10^{-2}	5.957×10^{-2}	7.432×10^{-3}	1.941×10^{-3}	3.225×10^{-5}
0.9	3.230×10^{-2}	7.721×10^{-2}	3.591×10^{-3}	2.691×10^{-3}	1.359×10^{-4}

where μ is a constant and k_2 is the thermal conductivity at T_2 . The problem which describes the process is:

$$\frac{d}{dx}\left(k \cdot \frac{dT}{dx}\right) = 0, \qquad x \in [0, L]$$
$$T(0) = T_1, \qquad T(L) = T_2.$$

Using the dimensionless variables:

$\theta = \dfrac{T - T_2}{T_1 - T_2}$, $y = \frac{x}{L}$, $\epsilon = \mu(T_1 - T_2) = \dfrac{k_1 - k_2}{k_2}$, we obtain the following problem:

$$\frac{d^2\theta}{dy^2}(1 + \epsilon\theta) + \epsilon\left(\frac{d\theta}{dy}\right)^2 = 0, \tag{17}$$

$$y \in [0, 1], \quad \theta(0) = 1, \quad \theta(1) = 0. \tag{18}$$

The exact solution of this equation is:

$$\theta_{exact} = \frac{\sqrt{1 - \epsilon(y-1)(\epsilon+2)} - 1}{\epsilon}.$$

Using the LSDQM we computed the following approximate solutions:
- 8^{th} order polynomial approximate solution for the problem (18-19): $\theta(\widetilde{y}) = -0.211473y^8 + 0.629685y^7 - 0.787099y^6 + 0.464842y^5 - 0.171033y^4 - 0.0340914y^3 - 0.140741y^2 -$

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$0.75009y + 1$, and
- 10^{th} order polynomial approximate solution $\theta(\tilde{y}) = -0.243307y^{10} + 0.979458y^9 - 1.71y^8 + 1.6329y^7 - 0.938876y^6 + 0.302677y^5 - 0.0846321y^4 - 0.0475601y^3 - 0.140633y^2 - 0.750019y + 1$.

In Table 4 we present the comparison between the solutions obtained by LSDQM and the solution obtained by Rajabi et all in ([14]) using HPM, Domairry et all in ([11]) using HAM and GAM and Caruntu and Bota in ([8]) using $SRMM$.

Table 4: Comparison of HPM, HAM,GAM, SRMM and LSDQ for $\epsilon = 1$

t	HPM	HAM	GAM	$SRMM8^{th}$ deg	$LSDQ8^{nd}$ deg	$LSDQ10^{th}$ deg
0.1	1.903×10^{-2}	8.659×10^{-5}	5.406×10^{-6}	5.692×10^{-7}	2.098×10^{-6}	4.440×10^{-7}
0.2	2.790×10^{-2}	9.110×10^{-5}	1.089×10^{-5}	2.598×10^{-6}	2.964×10^{-6}	1.710×10^{-7}
0.3	2.918×10^{-2}	3.066×10^{-4}	1.568×10^{-5}	1.241×10^{-6}	1.614×10^{-6}	2.203×10^{-7}
0.4	2.532×10^{-2}	1.320×10^{-3}	2.005×10^{-5}	4.794×10^{-6}	5.804×10^{-6}	2.737×10^{-7}
0.5	1.863×10^{-2}	3.013×10^{-3}	2.183×10^{-5}	3.750×10^{-7}	1.171×10^{-6}	2.613×10^{-7}
0.6	1.123×10^{-2}	5.239×10^{-3}	2.169×10^{-5}	5.500×10^{-6}	8.370×10^{-6}	2.799×10^{-7}
0.7	4.904×10^{-3}	7.529×10^{-3}	1.887×10^{-5}	4.652×10^{-7}	1.537×10^{-6}	9.978×10^{-7}
0.8	9.110×10^{-4}	8.911×10^{-3}	1.506×10^{-5}	6.420×10^{-6}	5.780×10^{-6}	8.774×10^{-7}
0.9	3.245×10^{-4}	7.550×10^{-3}	8.425×10^{-6}	2.698×10^{-6}	8.202×10^{-6}	1.638×10^{-6}

4 Conclusions

In the present paper we obtain analytical approximate solutions for several nonlinear heat transfer problems using the recently introduced Least Squares Differential Quadrature Method. Using the Least Squares Differential Quadrature Method one obtains the analytical solution of the problem, not only numerical solutions, fact which demonstrates the usefulness of the (LSDQM). The applications presented clearly illustrate the accuracy of the method.

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