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APPROXIMATE SOLUTIONS FOR RICCATI DIFFERENTIAL EQUATION OF FRACTIONAL ORDER USING THE LEAST SQUARES DIFFERENTIAL QUADRATURE METHOD

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Abstract

In the present paper we employ a recently introduced approximation method, namely the Least Squares Differential Quadrature Method (LSDQM), in order to compute analytical approximate polynomial solutions for several quadratic Riccati differential equation of fractional order. [1]

Keywords and phrases: *Riccati differential equation of fractional order, Least squares differential quadrature method (LSDQM)*

1 Introduction

In the last decades, fractional differential equations were the focus of intense research due to their importance in many real life applications like viscoelasticity ([1]), chemical engineering ([2]),fluid mechanics ([5]) or signal processing ([4]). In this paper we will consider the following class of Riccati differential equations of fractional order:

$$D^{\alpha}y(t) - A(t)y^2(t) - B(t)y(t) = f(t), \;\; 0 < \alpha \leq 1, \;\; t \in [0,1], \tag{1}$$

together with an initial condition of the type:

$$y(0) = k, \tag{2}$$

where $A(t)$, $B(t)$ and $C(t)$ are given real functions, k is a given real constant and $D^{\alpha}y(t)$ denotes Caputo's fractional derivative of order α:

$$D^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)} \cdot \int_0^t (t-\zeta)^{-(\alpha-n+1)} \cdot y^{(n)}(\zeta)d\zeta, \qquad n-1 < \alpha \leq n \tag{3}$$

[1]MSC (2010): 34K28, 45L05

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The Riccati differential equation of fractional order has been studied both numerically and analytically in numerous articles. Among the methods used to solve this equation we mention: Jacobi collocation method (JCM)([3]), Variational iteration method (VIM)([6]), modified homotopy perturbation method ($MHPM$) ([7]), optimal homotopy asymptotic method ($OHAM$) ([8]), Polynomial Least Squares Method ($PLSM$) ([9]).

2 The Least Squares Differential Quadrature Method (LSDQM)

The Least Squares Differential Quadrature Method (LSDQM) ([10]) allow us to compute approximate analytical polynomial solutions for fractional differential Riccati equations of the type (1). In order to obtain an approximate analytical polynomial solution for the equation (1), we will consider a numerical meshing of the interval $I = [0, 1]$ by means of a partition Δ_M consisting of $M + 1$ equidistant points:
$\Delta_M : \ 0 = t_0 < t_1 < t_2 < \cdots < t_{M-1} < t_M = 1.$
To the equation (1) we attach the following operator D:

$$D(y(t)) = D^{\alpha}y(t) - A(t)y^2(t) - B(t)y(t) - f(t). \tag{4}$$

We denote by $\tilde{y}(t)$ an approximate solution of the equation (1). By replacing in D the exact solution $y(t)$ with this approximate solution we obtain the *reminder* $\mathcal{R}$:

$$\mathcal{R}(t, \tilde{y}(t)) = D(\tilde{y}(t)), \qquad t \in [0, 1]. \tag{5}$$

Definition 1. *We call an* ϵ***-approximate*** *solution of the problem (1 - 2) related to the partition* Δ_M *an approximate polynomial solution which satisfies the following relations:*

$$\mathcal{R}(t_i, \tilde{y}(t_i)) < \epsilon, \qquad i = \overline{0, M}, \tag{6}$$

$$\tilde{y}(0) = k. \tag{7}$$

Definition 2. *We consider the sequence of polynomials:*

$$P_N(t) = \sum_{k=0}^{N} d_k t^k, \qquad d_k \in \mathbb{R}, \qquad k = \overline{0, N}. \tag{8}$$

We call the sequence of polynomials $P_N(t)$ ***convergent*** *to the solution of the problem (1 - 2) if:*

$$\lim_{N \to \infty} D(P_N(t)) = 0. \tag{9}$$

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We will compute ϵ - approximate polynomial solutions of the type:

$$T_N(t) = \sum_{k=0}^{N} \tilde{d}_k t^k, \tag{10}$$

with the initial condition: $T_N(0) = k$.

The constants $\tilde{d}_k$ are calculated by taking the following steps:

- From the initial condition we obtain $\tilde{d}_0$ as function of $\tilde{d}_1, \tilde{d}_2 \cdots \tilde{d_N}$ and replace them in the expression of $T_N(t)$ (which from now on will be a function of $\tilde{d}_1, \tilde{d}_2, \cdots, \tilde{d}_N$ only).
- We attach to the problem (1 - 2) the functional:

$$\mathcal{J}(\tilde{d}_1, \tilde{d}_2, \cdots, \tilde{d}_N) = \sum_{i=0}^{M} \mathcal{R}^2(t_i, T_N(t_i)). \tag{11}$$

- By minimizing the functional (11) we obtain the coefficients $\tilde{d}_1, \tilde{d}_2 \cdots \tilde{d_N}$.
- We replace the coefficients $\tilde{d}_1, \tilde{d}_2 \cdots \tilde{d_N}$.in the expression (10) and denote by $T_N^0(t) = \sum_{k=0}^{N} \tilde{d}_k t^k$, the analytical approximate polynomial solutions by LSDQM of the problem (1 - 2).

The following convergence theorem is satisfied:

Theorem 1. *The sequence of polynomials $T_N^0(t)$ satisfies the relations:*

$$\lim_{N\to\infty} \mathcal{R}^2(t_i, T_N^0(t_i)) = 0, \qquad i = \overline{0, M}. \tag{12}$$

Proof. Let $y(t)$ be an exact solution of the problem (1 - 2), which means from hypothesis that there exist a sequence of polynomials $P_N(t)$ with $P_N(t) = \sum_{k=0}^{N} d_k t^k$, $d_k \in \mathbb{R}$, $k = \overline{0, N}$ converging to $y(t)$: $\lim_{N\to\infty} P_N(t) = y(t)$, $\forall t \in I$.

We know that

$$\sum_{i=0}^{M} \mathcal{R}^2(t_i, T_N^0(t_i)) \leq \sum_{i=0}^{M} \mathcal{R}^2(t_i, P_N(t_i)),$$

hence

$$\lim_{N\to\infty} (\sum_{i=0}^{M} \mathcal{R}^2(t_i, T_N^0(t_i))) \leq \lim_{N\to\infty} (\sum_{i=0}^{M} \mathcal{R}^2(t_i, P_N(t_i))).$$

We conclude that $\lim_{N\to\infty} \mathcal{R}^2(t_i, T_N^0(t_i)) = 0, \qquad i = \overline{0, M}.$ □

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In that way we obtain the polynomial $T_N^0(t_i)$ which approximates the solution of the problem (1 - 2).

3 Applications

Application 1

We consider $A(t) = t^3$, $B(t) = -2t^4$, $f(t) = t^5 + 1$ and $k = 0$, obtaining the problem ([3]):

$$\begin{cases} D^\alpha y(t) - t^3 y^2(t) + 2t^4 y(t) = t^5 + 1 \\ y(0) = 0, \end{cases} \qquad for\ t \in\ [0,1]; \quad 0 < \alpha \leq 1 \tag{13}$$

Case $\alpha = 1$:

The exact solution of this problem for integer order $\alpha = 1$, is $y(t) = t$ ([3]).

Using the LSDQ method we computed a solution of the type (10):

$$P_1(t) = d_0 + d_1 \cdot t.$$

Taking into account the initial condition $P_1(0) = 0$, we obtain $d_0 = 0$, and the approximation becomes: $P_1(t) = d_1 \cdot t$.
The corresponding remainder (5) in this case is :

$$\mathcal{R}(t, P_1(t)) = -d_1^2 t^5 + 2d_1 t^5 + d_1 - t^5 - 1$$

and the functional$\mathcal{J}$ (11) is:

$$\mathcal{J}(\tilde{d_1}) = \tilde{d_1}^4 - 6\tilde{d_1}^3 + 13\tilde{d_1}^2 - 12\tilde{d_1} + 4.$$

We will find the value of $\tilde{d_1}$ by minimizing this functional. In order to do that we will first find the critical points as the solutions of the equation: $\mathcal{J}'(\tilde{d_1}) = 0$ and a simple computation shows that the minimum value is $\tilde{d_1} = 1$. Thus we are able to obtain the exact solution of the problem: $\tilde{y}(t) = t$.

Case $0 < \alpha \leq 1$:

Using the steps outlined in the previous section, we computed the following approximate solutions by LSDQ Method of the problem (12) :
- first order polynomial for $\alpha = 0.9$ $\tilde{y}(t) = 0.953415t$;
- first order polynomial for $\alpha = 0.8$ $\tilde{y}(t) = 0.923537t$;

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- first order polynomial for $\alpha = 0.7$ $\tilde{y}(t) = 0.905487t$;
These approximate analytical solutions by the Least Squares Differential Quadrature Method are presented the Fig. 1:

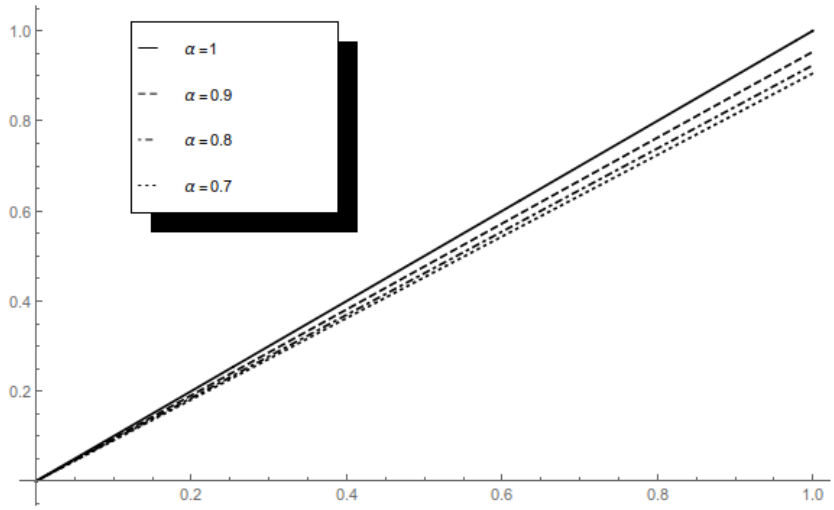

Figure 1: The LSDQM solution for Application 1 with $\alpha \in (0, 1]$

Application 2

Choosing $A(t) = -1$, $B(t) = 0$, $f(t) = 1$, and $k = 0$, the problem (1-2) becomes ([3]):

$$\begin{cases} D^{\alpha}y(t) + y^2(t) = 1 \\ y(0) = 0, \end{cases} \qquad for\ t \in\ [0,1]; \quad 0 < \alpha \leq 1 \tag{14}$$

The exact solution of this problem **for integer order** $\alpha = 1$ is given as: $y(t) = \dfrac{e^{2t}-1}{e^{2t}+1}$ ([3]). Following the LSDQM steps presented in the previous section, we compute a fifth degree polynomial approximate solution of the type (10):

$$P_5(t) = d_0 + d_1 \cdot t + d_2 \cdot t^2 + d_3 \cdot t^3 + d_4 \cdot t^4 + d_5 \cdot t^5.$$

Using the boundary condition we obtain $d_0 = 0$ and the remainder operator is:

$$\mathcal{R}(t, P_5(t)) = \left(d_1 t + d_2 t^2 + d_3 t^3 + d_4 t^4 + d_5 t^5\right)^2 + d_1 + 2d_2 t + 3d_3 t^2 + 4d_4 t^3 + 5d_5 t^4 - 1.$$

We consider the partition Δ_{100}: $0 = t_0 < t_1 < \cdots < t_{100} = 1$ with equidistant points $t_i = t_{i-1} + h$, where $h = 0.01$.

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Next we attach to the problem (14) the functional

$$\mathcal{J}(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{d}_4, \tilde{d}_5) = \sum_{i=0}^{100} \mathcal{R}^2(t_i, P_5(t_i))$$

(too long to be included here).
By minimizing this functional we find $\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{d}_4, \tilde{d}_5$. Replacing this values in the last expression of $P_5(x)$ we obtain the analytical approximate solution of the problem (14) by LSDQM as:

$$\tilde{y}(t) = -0.0169741t^5 + 0.187308t^4 - 0.42752t^3 + 0.0201815t^2 + 0.998592t.$$

We also solved the problem using a polynomial approximation of seventh degree and following the steps described above, we obtain the analytical approximate solution:

$$\tilde{y}(t) = 0.0218375t^7 - 0.122277t^6 + 0.226193t^5 - 0.0385853t^4 - \\ - 0.324675t^3 - 0.000934549t^2 + 1.00004t.$$

Making an approximation with a polynomial of ninth degree we obtain the following analytical approximate solution:

$$\tilde{y}(t) = -0.0106499t^9 + 0.0567198t^8 - 0.103159t^7 + 0.0253077t^6 + 0.125355t^5 + \\ + 0.00150336t^4 - 0.333491t^3 + 7.886787623886357 \times 10^{-6}t^2 + 1.0t$$

In Table 1 we present the comparison between the solutions obtained by Least Squares Differential Quadrature Method ($LSDQM$) and the solutions obtained by H. Singh and H. Sirvastava in ([3]) using the Jacobi collocation method (JCM), by B. Batiha et all in ([6]) using the Variational iteration method (VIM) and by Z. Odibat using a modified homotopy perturbation method ($MHPM$).

In the Figure 2 we present the error obtained for the approximation with nine degree polynomial for the solution $\tilde{y}(t)$ for $t \in [0, 1]$.

Using LSDQ Method we computed the following approximate solution for: $0 < \alpha \leq 1$, the approximate solution being of polynomial type of nine degree:

- for $\alpha = 0.9$, $\tilde{y}(t) = 22.895t^9 - 113.364t^8 + 237.882t^7 - 275.595t^6 + 192.552t^5 - 83.2439t^4 + 22.0942t^3 - 4.00524t^2 + 1.53857t$
- for $\alpha = 0.8$, $\tilde{y}(t) = 29.9235t^9 - 161.836t^8 + 371.86t^7 - 473.01t^6 + 363.877t^5 - 173.923t^4 + 51.5883t^3 - 10.0314t^2 + 2.29541t$

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Table 1: Comparison of absolute error from JCM, VIM, MHPM and LSDQ Method for $\alpha = 1$

t	JCM	VIM	$MHPM$	$LSDQ5^{th}$deg	$LSDQ7^{nd}$deg	$LSDQ9^{th}$deg
0.1	$4.57 \cdot 10^{-9}$	$5 \cdot 10^{-11}$	0	$1.59 \cdot 10^{-5}$	$1.44 \cdot 10^{-7}$	$1.23 \cdot 10^{-9}$
0.2	$9.74 \cdot 10^{-10}$	$4.39 \cdot 10^{-9}$	0	$2.43 \cdot 10^{-5}$	$1.71 \cdot 10^{-7}$	$2.14 \cdot 10^{-10}$
0.3	$3.71 \cdot 10^{-9}$	$1.56 \cdot 10^{-7}$	$1.10 \cdot 10^{-6}$	$1.41 \cdot 10^{-5}$	$5.6 \cdot 10^{-7}$	$3.78 \cdot 10^{-10}$
0.4	$1.29 \cdot 10^{-9}$	$1.97 \cdot 10^{-6}$	$5.01 \cdot 10^{-6}$	$2.32 \cdot 10^{-5}$	$2.55 \cdot 10^{-8}$	$3.52 \cdot 10^{-9}$
0.5	$1.93 \cdot 10^{-9}$	$1.38 \cdot 10^{-5}$	$3.91 \cdot 10^{-5}$	$3.95 \cdot 10^{-5}$	$2.55 \cdot 10^{-7}$	$1.71 \cdot 10^{-9}$
0.6	$2.74 \cdot 10^{-9}$	$6.61 \cdot 10^{-5}$	$1.93 \cdot 10^{-4}$	$1.82 \cdot 10^{-5}$	$1.80 \cdot 10^{-7}$	$1.10 \cdot 10^{-9}$
0.7	$4.32 \cdot 10^{-9}$	$2.43 \cdot 10^{-4}$	$7.37 \cdot 10^{-4}$	$1.58 \cdot 10^{-5}$	$4.27 \cdot 10^{-7}$	$4.08 \cdot 10^{-9}$
0.8	$2.43 \cdot 10^{-9}$	$7.35 \cdot 10^{-4}$	$2.33 \cdot 10^{-3}$	$2.18 \cdot 10^{-5}$	$1.14 \cdot 10^{-8}$	$2.38 \cdot 10^{-9}$
0.9	$3.59 \cdot 10^{-10}$	$1.91 \cdot 10^{-3}$	$6.37 \cdot 10^{36}$	$1.05 \cdot 10^{-5}$	$8.92 \cdot 10^{-8}$	$1.81 \cdot 10^{-9}$

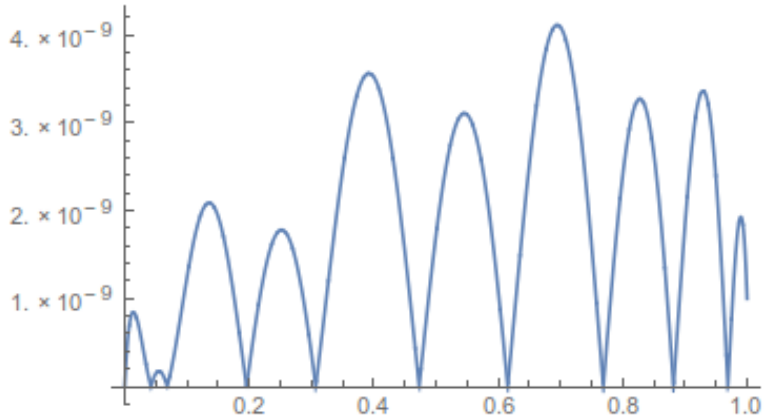

Figure 2: The absolute error corresponding to $\tilde{y}(t)$ for the case $\alpha = 1$ in Eq (14)

- for $\alpha = 0.7$, $\tilde{y}(t) = 19.9911t^9 - 147.74t^8 + 423.678t^7 - 637.806t^6 + 560.682t^5 - 298.769t^4 + 96.9138t^3 - 19.5875t^2 + 3.36745t$

In the Figure 3 we have plotted the approximate solutions for $\alpha = 0.7, 0.8, 0.9$ and 1.

Application 3

Our third application is the following Riccati differential equation of fractional order with initial condition:

$$\begin{cases} D^{\alpha}y(t) + y^2(t) - 2y(t) = 1 \\ y(0) = 0, \end{cases} \qquad for\ t \in\ [0,1]; \quad 0 < \alpha \leq 1 \qquad (15)$$

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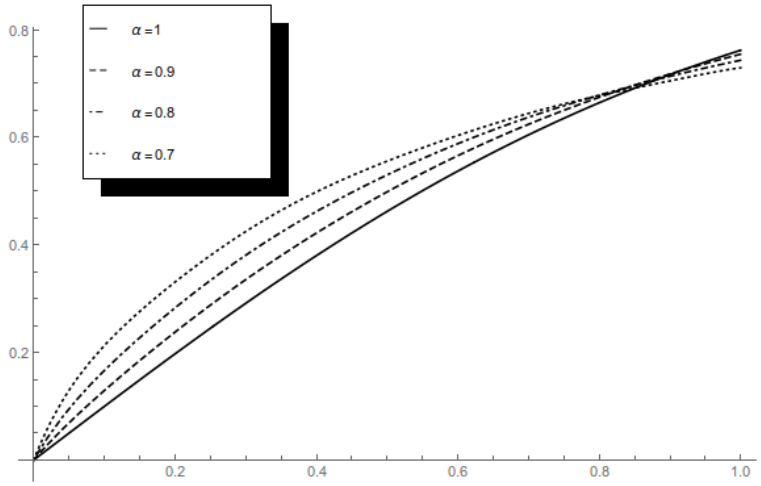

Figure 3: The LSDQM solution for Application 2 with $\alpha \in (0, 1]$

For $\alpha = 1$ the exact solution of the problem is ([3]):

$$y(t) = \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{1+\sqrt{2}}\right)\right) + 1.$$

In this case (equation (15) for $\alpha = 1$), by using LSDQM we computed the following approximate solution:
$\tilde{y}(t) = 0.225567t^9 - 1.37073t^8 + 2.94539t^7 - 2.49398t^6 + 0.64701t^5 - 0.640656t^4 + 0.380312t^3 + 0.996486t^2 + 1.0001t.$
Figure 4 presents the absolute error corresponding to our approximate solution $\tilde{y}(t)$ as the difference in absolute value between the approximate solution and the exact solution for the case $\alpha = 1$.

In Table 2 we present the comparison between the Least Squares Differential Quadrature Method ($LSDQM$) solution and the solutions obtained by H. Singh and H. Sirvastava in ([3]) using Jacobi collocation method (JCM), B. Batiha et all in ([6]) using Variational iteration method (VIM) and F. Mabood et all in ([8]) using optimal homotopy asymptotic method ($OHAM$).

Using LSDQ Method we computed the following approximate solution:

- for $\alpha = 0.9$, $\tilde{y}(t) = 21.6719t^9 - 106.446t^8 + 220.499t^7 - 251.061t^6 + 172.905t^5 - 75.3353t^4 + 19.9157t^3 - 1.93746t^2 + 1.5514t$
- for $\alpha = 0.8$, $\tilde{y}(t) = 38.1977t^9 - 195.105t^8 + 424.005t^7 - 512.343t^6 + 377.939t^5 - 174.904t^4 + 48.2756t^3 - 6.6356t^2 + 2.3775t$

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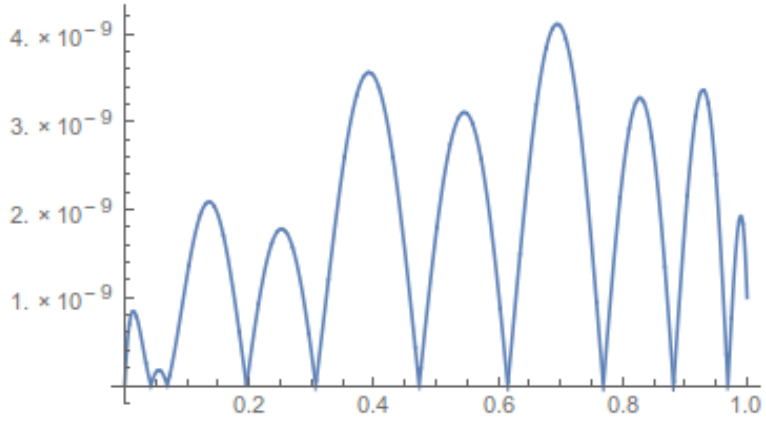

Figure 4: The absolute error corresponding to $\tilde{y}(t)$ for the case $\alpha = 1$ in Eq (15)

Table 2: Comparison of absolute error from JCM, VIM, OHAM and LSDQ Method for $\alpha = 1$

t	JCM	VIM	$OHAM$	$LSDQM$
0.1	$7.45 \cdot 10^{-7}$	$1.98 \cdot 10^{-8}$	$3.2 \cdot 10^{-5}$	$6.13 \cdot 10^{-8}$
0.2	$8.51 \cdot 10^{-7}$	$1.03 \cdot 10^{-6}$	$2.90 \cdot 10^{-4}$	$1.37 \cdot 10^{-6}$
0.3	$9.30 \cdot 10^{-7}$	$8.85 \cdot 10^{-6}$	$1.10 \cdot 10^{-3}$	$1.23 \cdot 10^{-7}$
0.4	$1.08 \cdot 10^{-6}$	$3.33 \cdot 10^{-51}$	$2.50 \cdot 10^{-3}0$	$4.75 \cdot 10^{-7}$
0.5	$1.14 \cdot 10^{-6}$	$7.26 \cdot 10^{-5}$	$4.40 \cdot 10^{-3}0$	$1.68 \cdot 10^{-6}$
0.6	$1.14 \cdot 10^{-6}$	$9.96 \cdot 10^{-5}$	$5.50 \cdot 10^{-3}0$	$5.63 \cdot 10^{-7}$
0.7	$1.21 \cdot 10^{-6}$	$8.84 \cdot 10^{-5}$	$5.51 \cdot 10^{-3}$	$2.90 \cdot 10^{-7}$
0.8	$1.04 \cdot 10^{-6}$	$1.54 \cdot 10^{-5}$	$3.80 \cdot 10^{-3}$	$1.48 \cdot 10^{-6}$
0.9	$1.13 \cdot 10^{-6}$	$4.95 \cdot 10^{-4}$	$3.20 \cdot 10^{-3}$	$2.91 \cdot 10^{-7}$
1	$4.84 \cdot 10^{-7}$	$3.47 \cdot 10^{-3}$	$3.40 \cdot 10^{-3}$	$8.37 \cdot 10^{-7}$

- for $\alpha = 0.7$, $\tilde{y}(t) = 32.9146t^9 - 197.034t^8 + 495.78t^7 - 685.108t^6 + 567.588t^5 - 286.392t^4 + 83.9888t^3 - 13.5326t^2 + 3.61622t$

In the Figure 4 we have plotted the approximate solution for different value of $\alpha = 0.7, 0.8, 0.9$ and 1.

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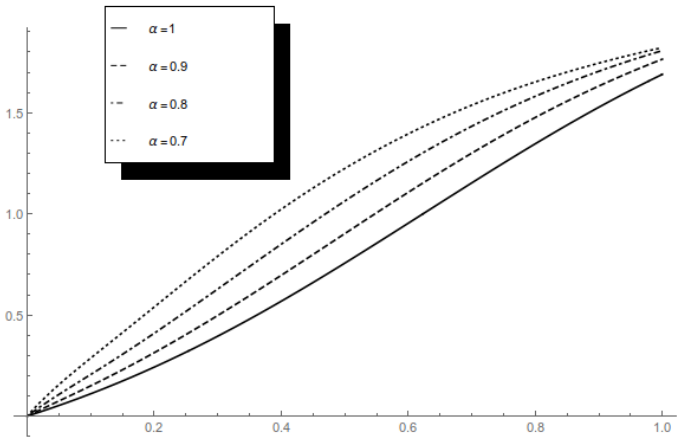

Figure 5: The LSDQM solution for Application 3 with $\alpha \in (0, 1]$

4 Conclusions

In the present paper we obtained analytical approximate solutions for several Riccati differential equations of fractional order using the Least Squares Differential Quadrature Method. Using the LSDQM one obtains the analytical solution of the equations, not only numerical solutions, fact which demonstrates the usefulness of the (LSDQM). The applications presented clearly illustrate good concordance between the LSDQM solutions and other approximate solutions.

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