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ON THE NORMAL FORM OF DOUBLE–HOPF BIFURCATION

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Abstract

A study on degenerate normal form of double-Hopf bifurcation is performed. This bifurcation is met in differential systems of dimension at least four and with minimum two independent parameters. We obtain bifurcation diagrams for amplitude system when one or two generic conditions are eliminated. [1]

Keywords and phrases: *dynamical systems, bifurcations, normal forms*

1 Introduction

In this work we aim to present a study related to normal form of double-Hopf bifurcation in generic four-dimensional differential system. A double-Hopf bifurcation arises in the following way. Let

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^4, \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2, \tag{1}$$

f smooth, be a four-dimensional differential system with two parameters. Assume that $x = 0$ is an equilibrium point of the system for all α with $|\alpha| = \sqrt{\alpha_1^2 + \alpha_1^2}$ small enough, that is, $f(0, \alpha) \equiv 0$; $x = 0$ stands for $x = (0, 0, 0, 0)$ and $\alpha = 0$ for $\alpha = (0, 0)$. The system (1) can be written as

$$\dot{x} = A(\alpha)\, x + F(x, \alpha) \tag{2}$$

where $F(x, \alpha) = O\left(|x|^2\right)$ is a smooth function denoting Taylor series with terms of order at least 2. Assume the matrix $A(\alpha)$ has two pairs of simple complex-conjugate eigenvalues λ_1, $\overline{\lambda}_1$, λ_2, $\overline{\lambda}_2$,

$$\lambda_1(\alpha) = \mu_1(\alpha) + i\omega_1(\alpha), \quad \lambda_2(\alpha) = \mu_2(\alpha) + i\omega_2(\alpha)$$

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for all sufficiently small $|\alpha|$, where $\mu_{1,2}(\alpha)$ and $\omega_{1,2}(\alpha)$ are smooth functions of α such that $\mu_1(0) = \mu_2(0) = 0$, respectively, $\omega_1(0) = \omega_{10} > 0$ and $\omega_2(0) = \omega_{20} > 0$.

When these conditions are satisfied, the system (1) undergoes a bifurcation at $\alpha = 0$, which is known as *double-Hopf* or *Hopf-Hopf bifurcation.* Some of the classical works dealing with qualitative properties of differential systems and, particularly, bifurcations, are [1], [2], [3], [5], [9], [14] and [16]. Of the first papers studying non-degenerate double-Hopf bifurcations are [6], [7], [8] and [10]. Recent results related to double-Hopf bifurcations can be found in [4], [11], [12], [13], [15], [17].

The paper is organized as it follows. After a short introduction, in section 2 we present the system to be studied in the new conditions. We transform it into a better form, while keeping the two systems locally topologically equivalent near the origin, and present the main properties of the system in the second form. In section 3, we study the system for the new conditions and obtain new bifurcation diagrams describing the behavior of the system. Conclusive remarks end the article.

2 General analysis

The system (2) can be successively transformed in new forms by several transformations as in [9], obtaining a form in complex variables (v_1, v_2). At this stage, the system is written in polar coordinates $v_1 = r_1 e^{i\varphi_1}$, $v_2 = r_2 e^{i\varphi_2}$, which give rise to a four-dimensional system in $(r_1, r_2, \varphi_1, \varphi_2)$. Truncating higher-order terms we obtain a system in which the variables (φ_1, φ_2) are decoupled by (r_1, r_2) and satisfy $\dot{\varphi}_i = \omega_i(0)$, $i = 1, 2$. Imposing now $\rho_i = r_i^2 > 0$, $i = 1, 2$, a new two-dimensional system is obtained, called *amplitude system*, given by

$$\begin{cases} \frac{d\rho_1}{d\tau} &= 2\rho_1\left[\mu_1 + p_{11}\rho_1 + p_{12}\rho_2 + p_{13}\rho_1\rho_2 + s_1\rho_2^2\right] \\ \frac{d\rho_2}{d\tau} &= 2\rho_2\left[\mu_2 + p_{21}\rho_1 + p_{22}\rho_2 + p_{23}\rho_1\rho_2 + s_2\rho_1^2\right] \end{cases}, \tag{3}$$

where $p_{ij} = p_{ij}(\mu)$ and $s_i = s_i(\mu)$, $i = 1, 2$, $j = 1, 2, 3$, are smooth functions depending on a parameter $\mu = (\mu_1, \mu_2)$, with $|\mu| = \sqrt{\mu_1^2 + \mu_2^2} < \varepsilon$ for some $\varepsilon > 0$ sufficiently small. In [9], using a rather complex transformation, the two terms in $\rho_1\rho_2$ from (3) are reduced to 0. We prefer to keep them but will show, as expected, they do not affect the local analysis of the system for $p_{ij}(0) \neq 0$, $i, j = 1, 2$, and $|\mu|$ sufficiently small.

Remark 2.1. *The first quadrant of the phase space, corresponding to* $\rho_1 \geq 0$ *and* $\rho_2 \geq 0$, *is invariant with respect to the flow of the system (3), because the axis*

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$\rho_1 = 0$ and $\rho_2 = 0$ are invariant to the flow. Thus, the system restricted to the first quadrant is well-defined and will be studied only in this region.

The system (3) with $p_{ij}(0) \neq 0$ for $i, j = 1, 2$ has been studied in [9] and a bifurcation-based analysis was presented. In this work we present a study for the system (3) when $p_{12}(\mu) p_{21}(\mu) = 0$ for all $|\mu|$ sufficiently small, $p_{11}(0) < 0$ and $p_{22}(0) < 0$. Due to the reasons explained above, the dynamics of the system will be considered only on the first quadrant $\rho_1 \geq 0$ and $\rho_2 \geq 0$.

Consider further the transformation $(\rho_1, \rho_2) \longmapsto (\xi_1, \xi_2)$ given by

$$\xi_1 = -p_{11}(\mu)\rho_1 \text{ and } \xi_2 = -p_{22}(\mu)\rho_2. \tag{4}$$

It is well defined and nonsingular for all $|\mu|$ small enough because $p_{11}(0) \neq 0$ and $p_{22}(0) \neq 0$. By changes (4) and $t = 2\tau$, the system (3) is locally topologically equivalent near the origin for all sufficiently small $|\mu|$ to

$$\begin{cases} \frac{d\xi_1}{dt} &= \xi_1\left(\mu_1 - \xi_1 - \theta\xi_2 + M\xi_1\xi_2 + N\xi_2^2\right) \\ \frac{d\xi_2}{dt} &= \xi_2\left(\mu_2 - \delta\xi_1 - \xi_2 + S\xi_1\xi_2 + P\xi_1^2\right) \end{cases}, \tag{5}$$

where $\theta = \frac{p_{12}}{p_{22}}$, $M = \frac{p_{13}}{p_{11}p_{22}}$, $N = \frac{s_1}{p_{22}^2}$, $\delta = \frac{p_{21}}{p_{11}}$, $S = \frac{p_{23}}{p_{11}p_{22}}$ and $P = \frac{s_2}{p_{11}^2}$, with $\theta = \theta(\mu)$, $\delta = \delta(\mu)$, $M = M(\mu)$, $N = N(\mu)$, $S = S(\mu)$ and $P = P(\mu)$ are smooth functions of their arguments.

The new system (5) inherits the property of being invariant with respect to the first quadrant $\xi_1 \geq 0$ and $\xi_2 \geq 0$, and the transformation (4) maps the first quadrant of the system (3) into the first quadrant of (5). Thus, the study of the new system (5) in the first quadrant is well-defined.

Remark 2.2. *The case $p_{11}(0) > 0$ and $p_{22}(0) > 0$ reduces to the previous case by a change in (3) of the form*
$\xi_1 = p_{11}\rho_1$, $\xi_2 = p_{22}\rho_2$, $t = 2\tau$ and $(\xi_1, \xi_2) \longmapsto (-\xi_1, -\xi_2)$.

Assume first $\theta(0)\delta(0) \neq 0$. Thus, $p_{ij}(0) \neq 0$, for all $i, j = 1, 2$. The dynamics of the system (5) is known in this case. More exactly, when $\theta(0)\delta(0) - 1 \neq 0$, $\theta(0)\delta(0) \neq 0$ and $p_{11}(0) p_{22}(0) > 0$, it has been shown in [9] that the system (5) is locally topologically equivalent near the origin for all $|\mu|$ sufficiently small to the system

$$\begin{cases} \frac{d\xi_1}{dt} &= \xi_1(\mu_1 - \xi_1 - \theta(0)\xi_2) \\ \frac{d\xi_2}{dt} &= \xi_2(\mu_2 - \delta(0)\xi_1 - \xi_2) \end{cases}. \tag{6}$$

In order to introduce the reader to specific methods of studying systems of type (5), we want to present in this section the main properties of the system (5). Bifurcation diagrams for (6) are presented in [9].

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Since in what follows, the expressions of $\theta = \theta(\mu)$, $\delta = \delta(\mu)$, $M = M(\mu)$, $N = N(\mu)$, $S = S(\mu)$ and $P = P(\mu)$ are needed only at $\mu = 0$, in order to save symbols, we denote by $\theta(0) = \theta$, $\delta(0) = \delta$ and so on.

The system (5) has three equilibria $O(0,0)$, $E_1(\mu_1, 0)$ and $E_2(0, \mu_2)$ for all $|\mu|$ sufficiently small. Another equilibrium $E_3(\xi_1, \xi_2)$ exists for $|\mu|$ sufficiently small, where

$$\begin{cases} \xi_1 = \frac{-\mu_1}{\theta\delta-1}(1 + O(|\mu|)) + \frac{\theta\mu_2}{\theta\delta-1}(1 + O(|\mu|)) \\ \xi_2 = \frac{\delta\mu_1}{\theta\delta-1}(1 + O(|\mu|)) + \frac{-\mu_2}{\theta\delta-1}(1 + O(|\mu|)) \end{cases}, \tag{7}$$

provided that

$$\theta\delta - 1 \neq 0. \tag{8}$$

$O\left(|\mu|^k\right) = \sum_{i+j\geq k} c_{ij}\mu_1^i\mu_2^j$ denotes a Taylor rest of order $k \geq 1$. The existence of E_3 is based on the Implicit Function Theorem applied to the algebraic system

$$\mu_1 - \xi_1 - \theta\xi_2 + M\xi_1\xi_2 + N\xi_2^2 = 0 \text{ and } \mu_2 - \delta\xi_1 - \xi_2 + S\xi_1\xi_2 + P\xi_1^2 = 0. \tag{9}$$

The four equilibrium points are well-defined whenever their coordinates are non-negative. Two bifurcation curves determine when E_3 is born or vanishes, namely

$$T_1 = \left\{(\mu_1, \mu_2) \in \mathbb{R}^2 \mid \mu_1 = \theta\mu_2 + O\left(\mu_2^2\right), \mu_2 > 0\right\} \tag{10}$$

and

$$T_2 = \left\{(\mu_1, \mu_2) \in \mathbb{R}^2 \mid \mu_2 = \delta\mu_1 + O\left(\mu_1^2\right), \mu_1 > 0\right\}. \tag{11}$$

On T_1, E_3 coincides to $E_2(0, \mu_2)$ while on T_2 to $E_1(\mu_1, 0)$; we call E_3 *trivial* in these cases, otherwise nontrivial.

Remark 2.3. *The eigenvalues of the first three equilibria O and $E_{1,2}$ are, respectively, $\lambda_{1,2}^O = \mu_{1,2}$, $\lambda_1^{E_1} = -\mu_1$ and $\lambda_2^{E_1} = -\delta\mu_1 + \mu_2 + P\mu_1^2$, and $\lambda_1^{E_2} = -\mu_2$ and $\lambda_2^{E_2} = \mu_1 - \theta\mu_2 + N\mu_2^2$.*

The characteristic polynomial of the system (5) at an equilibrium point (ξ_1, ξ_2) of type E_3, i.e. satisfying (9), has the form $P(\lambda) = \lambda^2 - 2p\lambda + B$ where

$$p = -\frac{1}{2}(\xi_1 + \xi_2) + \frac{1}{2}(M + S)\xi_1\xi_2 \text{ and } B = -\xi_1\xi_2(\theta\delta - 1 + O(|\xi|)). \tag{12}$$

It follows immediately from (12) that the product of the eigenvalues $\lambda_{1,2}$ of E_3 satisfy $\lambda_1\lambda_2 = B$, namely

$$\lambda_1\lambda_2 = -\frac{1}{\theta\delta - 1}(-\mu_1 + \theta\mu_2)(\delta\mu_1 - \mu_2)(1 + O(|\mu|)). \tag{13}$$

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Moreover, if $\lambda_{1,2}$ are of the form $\lambda_{1,2} = p \pm \sqrt{q}$ then

$$p = -\frac{1}{2(\theta\delta - 1)}\left(-\mu_1 + \theta\mu_2 + \delta\mu_1 - \mu_2\right) \text{ and } q = \frac{1}{4(\theta\delta-1)^2}\left(a\mu_1^2 + 2b\mu_1\mu_2 + c\mu_2^2\right) \tag{14}$$

in their lowest terms, where $a = (\delta+1)^2 - 4\theta\delta^2$, $b = 2\theta^2\delta^2 + \theta\delta - \theta - \delta - 1$ and $c = (\theta+1)^2 - 4\theta^2\delta$. In general,

$$B = -\xi_1\xi_2\left(\theta\delta - 1 + m_{11}\xi_1 + (M - 2N\delta - S\theta)\xi_2 + 2MP\xi_1^2 + 4NP\xi_1\xi_2 + 2NS\xi_2^2\right), \tag{15}$$

where $m_{11} = S - M\delta - 2P\theta$ and

$$q = \frac{1}{4}(\xi_1 - \xi_2)^2 + \frac{1}{4}\xi_1\xi_2\left(4\theta\delta + c_{11}\xi_1 + c_{12}\xi_2 + 8MP\xi_1^2 + c_{13}\xi_1\xi_2 + 8NS\xi_2^2\right) \tag{16}$$

where $c_{11} = 2S - 2M - 4M\delta - 8P\theta$, $c_{12} = 2M - 2S - 8N\delta - 4S\theta$ and $c_{13} = 16NP + (M+S)^2$. At $\xi_2 = \xi_1$, q becomes

$$q = \theta\delta\xi_1^2 + \frac{1}{4}(c_{11} + c_{12})\xi_1^3 + \frac{1}{4}(8MP + 8NS + c_{13})\xi_1^4. \tag{17}$$

We state in the following a result concerning the sign of p when $\theta\delta - 1 \neq 0$, which will be used later on to show the non-existence of Hopf bifurcation at E_3.

Lemma 2.4. *If $\theta\delta - 1 \neq 0$, then $p(\mu_1, \mu_2) < 0$ whenever E_3 exists (i.e in the first quadrant) for all $|\mu|$ sufficiently small.*

Proof. If $\theta\delta - 1 > 0$, $E_3(\xi_1, \xi_2)$ is well-defined (i.e. exists in the first quadrant of the phase space) and non-trivial when (μ_1, μ_2) lies in the region

$$R_1 = \left\{(\mu_1, \mu_2) \in \mathbb{R}^2 \mid -\mu_1 + \theta\mu_2 > 0, \delta\mu_1 - \mu_2 > 0, |\mu| < \varepsilon\right\},$$

while, if $\theta\delta - 1 < 0$, when (μ_1, μ_2) lies in

$$R_2 = \left\{(\mu_1, \mu_2) \in \mathbb{R}^2 \mid -\mu_1 + \theta\mu_2 < 0, \delta\mu_1 - \mu_2 < 0, |\mu| < \varepsilon\right\},$$

for some $\varepsilon > 0$ small enough. Denote by H the curve $p = 0$, that is,

$$H = \left\{(\mu_1, \mu_2) \in \mathbb{R}^2, p(\mu_1, \mu_2) = 0, |\mu| < \varepsilon\right\}.$$

When $\theta\delta - 1 \neq 0$, $\theta \neq 1$ and $\delta \neq 1$, it follows that H becomes

$$H_1 = \left\{(\mu_1, \mu_2), \mu_2 = -\mu_1\frac{\delta - 1}{\theta - 1}(1 + O(\mu_1))\right\}.$$

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In finding H_1, only the linear terms in ξ_1 and ξ_2 were needed from the expression of p given in (12), thus, when $|\mu|$ is small enough, we can approximate $p = -\frac{1}{2}(\xi_1 + \xi_2)$. It follows that $p < 0$ on $R_{1,2}$ because $\xi_{1,2} > 0$ on $R_{1,2}$.

It remains to study the cases $\theta = 1$, respectively, $\delta = 1$. Assume further $\theta\delta - 1 < 0$. A similar proof can be obtained for $\theta\delta - 1 > 0$.

Consider the first case $\theta = 1$ and $\delta < 1$; $\delta \neq 0$. It follows from (12) that p is of the form $p = \frac{1}{2}\left(-\mu_1 + k_1\mu_2^2\right)$, while H becomes

$$H_2 = \left\{(\mu_1, \mu_2), \mu_1 = k_1\mu_2^2\left(1 + O(\mu_2)\right)\right\}, \tag{18}$$

where we assume $k_1 \neq 0$. One can show $H_2 \nsubseteq R_2$, thus $p(\mu)$ keeps constant sign on R_2. Indeed, assuming that there exists a point $(\mu_1, \mu_2) \in H_2 \cap R_2$, we obtain $\mu_1 = k_1\mu_2^2$, $\mu_2(1 - \mu_2 k_1) < 0$ and $\delta k_1\mu_2^2 - \mu_2 = -\mu_2(1 - \delta\mu_2 k_1) < 0$ with $|\mu|$ sufficiently small, which lead to the contradiction $\mu_2 < 0$ and $\mu_2 > 0$.

If $\delta < 0$, then $(\mu_1, 0) \in R_2$ for $\mu_1 > 0$ which leads to $p(\mu_1, 0) = -\frac{1}{2}\mu_1 < 0$, thus, $p(\mu_1, \mu_2) < 0$ on R_2.

If $0 < \delta < 1$, then any point $(\mu_1, \mu_2) \in R_2$ satisfies $\mu_2 < \mu_1 < \frac{1}{\delta}\mu_2$ which lead to $\mu_2(1 - \delta) > 0$, that is, $\mu_2 > 0$, respectively, $\mu_1 > \mu_2 > 0$. Therefore, $R_2 \subset \{\mu_1 > 0, \mu_2 > 0\}$. A point $(\mu_1, \mu_2) \in R_2$ if and only if $\mu_2 = c_0\mu_1$ where $0 < \delta < c_0 < 1$ and $\mu_1 > 0$. For such a point we have $p = \frac{1}{2}\left(-1 + k_1c_0^2\mu_1\right)\mu_1 < 0$ for $|\mu|$ sufficiently small. It follows that $p(\mu_1, \mu_2) < 0$ on R_2.

Consider the second possibility, $\delta = 1$ and $\theta < 1$; $\theta \neq 0$. Then

$$p = \frac{1}{2}\left(-\mu_2 + k_2\mu_1^2\right)$$

by (12), while H becomes

$$H_3 = \left\{(\mu_1, \mu_2), \mu_2 = k_2\mu_1^2\left(1 + O(\mu_1)\right)\right\},$$

where we assume $k_2 \neq 0$. One shows similarly that $H_2 \nsubseteq R_2$. Indeed, if there exists a point $(\mu_1, \mu_2) \in H_2 \cap R_2$ then, $\mu_2 = k_2\mu_1^2$, $-\mu_1(1 - \theta k_2\mu_1) < 0$ and $\mu_1(1 - k_2\mu_1) < 0$. Since $|\mu|$ is sufficiently small, these lead to the contradiction $\mu_1 > 0$ and $\mu_1 < 0$. Thus, $H_2 \nsubseteq R_2$ and p keeps constant sign on R_2.

If $\theta < 0$, then $(0, \mu_2) \in R_2$ for $\mu_2 > 0$ which leads to $p(0, \mu_2) = -\frac{1}{2}\mu_2 < 0$, thus, $p(\mu_1, \mu_2) < 0$ on R_2.

If $0 < \theta < 1$, then any point $(\mu_1, \mu_2) \in R_2$ satisfies $\mu_1 < \mu_2 < \frac{1}{\theta}\mu_1$ which lead to $\mu_1(1 - \theta) > 0$, that is, $\mu_1 > 0$, respectively, $\mu_2 > \mu_1 > 0$. Therefore, $R_2 \subset \{\mu_1 > 0, \mu_2 > 0\}$. A point $(\mu_1, \mu_2) \in R_2$ if and only if $\mu_2 = b_0\mu_1$ where $1 < b_0 < \frac{1}{\theta}$ and $\mu_1 > 0$. For such a point we have $p = \frac{1}{2}(-b_0 + k_2\mu_1)\mu_1 < 0$ for $|\mu|$ sufficiently small. It follows that $p(\mu_1, \mu_2) < 0$ on R_2. □

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Theorem 2.5. *1) If $\theta\delta - 1 > 0$, the equilibrium E_3 is a saddle.*

2) If $\theta\delta - 1 < 0$, the equilibrium E_3 is stable (node or focus). Moreover, when $0 < \theta\delta < 1$, E_3 is a stable node whenever is nontrivial.

3) No Hopf bifurcation can occur at E_3 when $\theta\delta - 1 \neq 0$.

Proof. 1) The eigenvalues of $E_3\left(\frac{-\mu_1+\theta\mu_2}{\theta\delta-1}, \frac{\delta\mu_1-\mu_2}{\theta\delta-1}\right)$ in their lowest terms satisfy (13) and $\lambda_1 + \lambda_2 = 2p$.

Assume first $\theta\delta - 1 > 0$. Then a nontrivial E_3 exists when (μ_1, μ_2) lies in the region R_1, where $\lambda_1\lambda_2 < 0$. The eigenvalues λ_1, λ_2 are real numbers on R_1 because, otherwise, if $\lambda_{1,2} = p \pm i\omega$, then $\lambda_1\lambda_2 = p^2 + \omega^2 > 0$ which contradicts $\lambda_1\lambda_2 < 0$. Thus, E_3 is a saddle point with $q > 0$. This result may also be obtained from $\lambda_1\lambda_2 = -\xi_1\xi_2\left(\theta\delta - 1 + O\left(|\xi|\right)\right)$ because $\xi_1 > 0$ and $\xi_2 > 0$ are sufficiently small for $|\mu|$ small, that is, $\theta\delta - 1 + O\left(|\xi|\right) > 0$. By (17), the result remains valid at $\xi_2 = \xi_1$.

2) In the second case $\theta\delta - 1 < 0$, a nontrivial E_3 exists on the region R_2 where $\lambda_1\lambda_2 > 0$ by (12) and $\lambda_1 + \lambda_2 = 2p < 0$ by Lemma 2.4. It follows that $\lambda_{1,2} < 0$ if $q > 0$, respectively, $Re\left(\lambda_{1,2}\right) < 0$ if $q < 0$. Thus, E_3 is stable (node or focus) whenever $\theta\delta - 1 < 0$.

Assume further $0 < \theta\delta < 1$. The equation $q = 0$ in variable μ_1 has the discriminant $\Delta = b^2 - ac$ given by

$$\Delta = 4\theta\delta\left(\theta\delta - 1\right)^3\mu_2^2 < 0.$$

Thus, the sign of q is given by the sign of a, which is always strictly positive in this case. Indeed, if $\theta < 0$ and $\delta < 0$ then $a > 0$, while $a > (\delta - 1)^2 \geq 0$ if $\delta > 0$ and $\theta > 0$. Therefore, $q > 0$ which, in turn, implies that $\lambda_{1,2}$ are real numbers which satisfy $\lambda_1 + \lambda_2 < 0$ and $\lambda_1\lambda_2 > 0$ on R_2. It follows that $\lambda_{1,2} < 0$ and E_3 is a stable node. By (17), the result remains valid at $\xi_2 = \xi_1$ since $q = \theta\delta\xi_1^2\left(1 + O\left(|\xi|\right)\right)$ and $|\xi|$ is small enough to have $1 + O\left(|\xi|\right) > 0$.

3) From Lemma 2.4, it follows that $p \neq 0$ whenever E_3 exists and is nontrivial which, in turn, implies that the eigenvalues $\lambda_{1,2}$ cannot be purely complex of the form $\pm i\omega_0$ with $\omega_0 > 0$, thus, a Hopf bifurcation cannot occur at E_3. It implies that period orbits emerging from Hopf bifurcation do not exist around E_3. □

Remark 2.6. *Since the above analysis did not use the terms with the coefficients M, N, P and S from (5), it follows that the system (5) is locally topologically equivalent near the origin for all $|\mu|$ sufficiently small to the system (6).*

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3 Analysis and bifurcation diagrams at $\theta(\mu)\,\delta(\mu) = 0$

3.1 The case $\theta(\mu) = 0$ and $\delta(0) \neq 0$

Assume first $\theta(\mu) = 0$ for all $|\mu|$ sufficiently small and $\delta(0) \overset{not}{=} \delta \neq 0$. Then $E_3(\xi_1, \xi_2)$ exists and has its coordinates in their lowest terms given $\xi_1 = \mu_1 + N\mu_2^2$ and $\xi_2 = \mu_2 - \delta\mu_1$, provided that $N(0) \overset{not}{=} N \neq 0$. The following result describes the behavior of E_3.

Theorem 3.1. *If $\theta(\mu) = 0$ for all $|\mu|$ sufficiently small, $\delta(0) \neq 0$ and $N(0) \neq 0$, E_3 is an attractor (node or focus) whenever it exists for all $|\mu|$ sufficiently small.*

Proof. A nontrivial E_3 is well-defined when (μ_1, μ_2) lies in the region

$$R_3 = \left\{(\mu_1, \mu_2) \in \mathbb{R}^2 \mid \mu_1 + N\mu_2^2 > 0, \mu_2 - \delta\mu_1 > 0\right\}$$

where

$$p = -\frac{1}{2}\left(\mu_1 + N\mu_2^2 + \mu_2 - \delta\mu_1\right) < 0$$

if $\delta \neq 1$ and its eigenvalues satisfy $\lambda_1^{E_3}\lambda_2^{E_3} = \xi_1\xi_2\left(1 + O\left(|\xi|\right)\right) > 0$. Thus, E_3 is an attractor (node or focus) whenever $\delta \neq 1$. At $\delta = 1$, p is of the form

$$p = -\frac{1}{2}\mu_2\left(1 + O\left(|\mu|\right)\right) + h_1\mu_1^2\left(1 + O\left(|\mu|\right)\right).$$

Assume $h_1 \neq 0$. Therefore, from the Implicit Function Theorem, $p = 0$ is a curve of the form

$$H_4 = \left\{(\mu_1, \mu_2), \mu_2 = 2h_1\mu_1^2\left(1 + O\left(\mu_1\right)\right)\right\}.$$

We notice that $H_4 \cap R_3 = \emptyset$. Indeed, assuming that there exists a point $(\mu_1, \mu_2) \in H_4 \cap R_3$, then $\mu_1\left(1 + O\left(\mu_1\right)\right) > 0$ and $-\mu_1\left(1 + O\left(\mu_1\right)\right) > 0$, which is a contradiction for a fixed μ_1 with $|\mu_1|$ small enough. Thus, $p(\mu)$ keeps constant sign on R_3.

Assume first $N > 0$. Then $(0, \mu_2) \in R_3$ for all $\mu_2 > 0$ and $p(0, \mu_2) =$
$= -\frac{1}{2}\mu_2\left(1 + O\left(|\mu|\right)\right) < 0$, thus, $p(\mu_1, \mu_2) < 0$ for all $(\mu_1, \mu_2) \in R_3$.

If $N < 0$, take a point of the form $\left(-2N\mu_2^2, \mu_2\right) \in R_3$ for all $\mu_2 > 0$ sufficiently small. Then $p\left(-2N\mu_2^2, \mu_2\right) = -\frac{1}{2}\mu_2\left(1 + O\left(|\mu|\right)\right) < 0$, thus, $p(\mu_1, \mu_2) < 0$ for all $(\mu_1, \mu_2) \in R_3$.

While typically $h_1 \neq 0$, when $h_1 = 0$ we can write p in the form

$$p = -\frac{1}{2}\mu_2\left(1 + O\left(|\mu|\right)\right) + h_n\mu_1^n\left(1 + O\left(|\mu|\right)\right)$$

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for some $n \geq 2$ with $h_n \neq 0$. However, this does not change the proof and conclusion. □

Remark 3.2. *Typically E_3 is a node when*

$$\theta(\mu)\,\delta(\mu) = 0$$

because $q = \frac{1}{4}(\xi_1 - \xi_2)^2 + O\left(|\xi|^3\right)$.

Remark 3.3. *At $\theta(\mu) = 0$ and $\delta \neq 0$, the first three equilibria O and $E_{1,2}$ still exist with the corresponding eigenvalues*

$$\lambda_{1,2}^{O} = \mu_{1,2},$$

$\lambda_1^{E_1} = -\mu_1$ *and* $\lambda_2^{E_1} = -\delta\mu_1 + \mu_2$, *respectively,* $\lambda_1^{E_2} = -\mu_2$ *and* $\lambda_2^{E_2} = \mu_1 + N\mu_2^2$.

Two bifurcation curves determine the existence of E_3, namely T_2 (11) given by $\mu_2 = \delta\mu_1(1 + O(\mu_1))$, $\mu_1 > 0$, and

$$C_1 = \left\{(\mu_1, \mu_2) \in \mathbb{R}^2 \mid \mu_1 = -N\mu_2^2 + O\left(\mu_2^3\right)\right\}. \tag{19}$$

Different to T_2, the curve C_1 is tangent to the μ_2-axis at O. It is a parabola when $|\mu|$ is sufficiently small. Taking into account the signs of δ and N, four different bifurcation diagrams arise in this case, Figure 1. The phase portraits corresponding to the eight regions $1-8$ from the bifurcation diagrams $D1-D12$ are depicted in Figure 4.

The next table summarizes the dynamics of the four equilibria on different regions of the parametric plane $\mu_1\mu_2$ as they appear on the bifurcation diagrams for $\theta(\mu) = 0$ and $\delta(0) \neq 0$.

	1	2	3	4	5	6	7	8
O	s	r	r	r	s	a	s	s
E_1	$-$	s	s	a	a	$-$	$-$	s
E_2	a	a	s	s	$-$	$-$	s	$-$
E_3	$-$	$-$	a	$-$	$-$	$-$	a	a

Table 1: *Types of the four equilibria at $\theta(\mu)\delta(\mu) = 0$. The abbreviations are the followings: a for attractor, s for saddle and r for repeller.*

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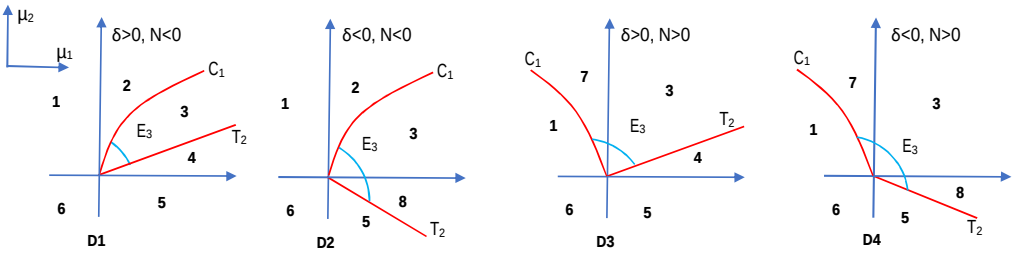


Figure 1: Bifurcation diagrams corresponding to $\theta(\mu) = 0$ and $\delta(0) \neq 0$.

3.2 The case $\theta(0) \neq 0$ and $\delta(\mu) = 0$

Secondly, assume $\theta(0) \overset{not}{=} \theta \neq 0$ and $\delta(\mu) = 0$ for all $|\mu|$ sufficiently small. Then E_3 in its lowest terms read $E_3(\xi_1^*, \xi_2^*)$ with $\xi_1^* = \mu_1 - \theta\mu_2$ and $\xi_2^* = \mu_2 + P\mu_1^2$. It is nontrivial on the region

$$R_4 = \left\{(\mu_1, \mu_2) \mid \mu_1 - \theta\mu_2 > 0, \mu_2 + P\mu_1^2 > 0\right\}$$

where $\lambda_1\lambda_2 = \xi_1\xi_2(1 + O(|\xi|)) > 0$ and $\lambda_1 + \lambda_2 = 2p < 0$ if $\theta \neq 1$, where p in its lowest terms is $p = -\frac{1}{2}\left(\mu_1 - \theta\mu_2 + \mu_2 + P\mu_1^2\right)$.

One shows similarly to the previous case that $p < 0$ for $\theta = 1$ as well. In conclusion,we can write the following theorem.

Theorem 3.4. *If $\delta(\mu) = 0$ for all $|\mu|$ sufficiently small, $\theta(0) \neq 0$ and $P(0) \neq 0$, E_3 is an attractor (node or focus) whenever it exists for all $|\mu|$ sufficiently small.*

The existence of E_3 is determined by T_1 (10) given by $\mu_1 = \theta\mu_2(1 + O(\mu_2))$, $\mu_2 > 0$, and a curve C_2 tangent to the μ_1-axis at O, given by

$$C_2 = \left\{(\mu_1, \mu_2) \in \mathbb{R}^2 \mid \mu_2 = -P\mu_1^2(1 + O(\mu_1))\right\}. \tag{20}$$

Remark 3.5. *The eigenvalues of the first three equilibria are $\lambda_{1,2}^O = \mu_{1,2}$, $\lambda_1^{E_1} = -\mu_1$ and $\lambda_2^{E_1} = \mu_2 + P\mu_1^2$, respectively, $\lambda_1^{E_2} = -\mu_2$ and $\lambda_2^{E_2} = \mu_1 - \theta\mu_2$.*

Four different bifurcation diagrams emerge in this case as well, Figure 2, depending on the signs of θ and P.

3.3 The case $\theta(\mu) = 0$ and $\delta(\mu) = 0$

Finally, assume $\theta(\mu) = \delta(\mu) = 0$. Then $E_3(\xi_1^*, \xi_2^*)$ has its coordinates $\xi_1^* = \mu_1 + N\mu_2^2$ and $\xi_2^* = \mu_2 + P\mu_1^2$, where $N \overset{def}{=} N(0) \neq 0$ and $P \overset{def}{=} P(0) \neq 0$.

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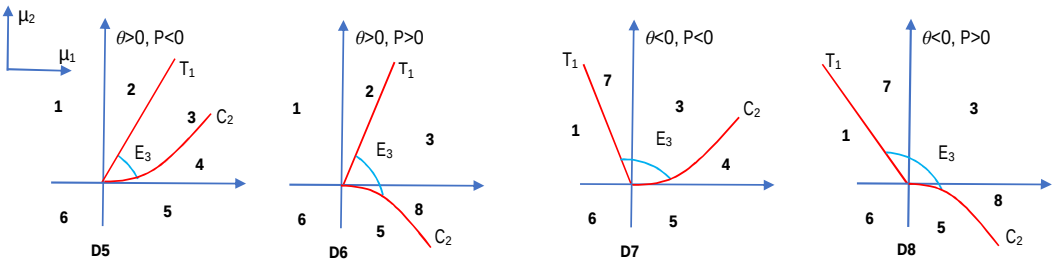


Figure 2: Bifurcation diagrams corresponding to $\delta(\mu) = 0$ and $\theta(0) \neq 0$.

E_3 exists and is nontrivial on the region stable

$$R_5 = \left\{(\mu_1, \mu_2) \in \mathbb{R}^2 \mid \mu_1 + N\mu_2^2 > 0, \mu_2 + P\mu_1^2 > 0\right\}$$

where

$$p = -\frac{1}{2}(\mu_1 + \mu_2) + O\left(|\mu|^2\right).$$

It follows that $p < 0$ for $|\mu|$ sufficiently small, thus, E_3 is an attractor. Typically, E_3 is a node since $q = (\mu_1 + \mu_2)^2 \geq 0$ in its lowest terms.

Remark 3.6. *The eigenvalues of the first three equilibria are* $\lambda_{1,2}^{O} = \mu_{1,2}$, $\lambda_1^{E_1} = -\mu_1$ *and* $\lambda_2^{E_1} = \mu_2 + P\mu_1^2$, *respectively,* $\lambda_1^{E_2} = -\mu_2$ *and* $\lambda_2^{E_2} = \mu_1 + N\mu_2^2$.

Depending on the signs of N and P, four different bifurcations diagrams are obtained, Figure 3. The bifurcation diagrams depicted in Figures 1-3 corresponding to the three cases arising from $\theta(\mu)\,\delta(\mu) = 0$, are different in terms of the existing bifurcation curves in the three cases but the order of different regions in the diagrams from different cases may coincide.

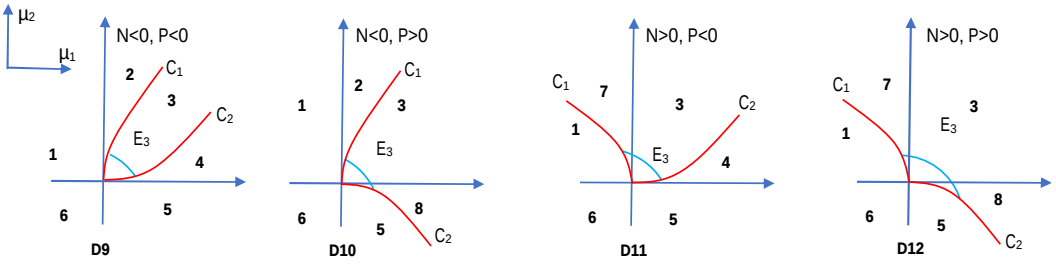


Figure 3: Bifurcation diagrams corresponding to $\theta(\mu) = 0$ and $\delta(\mu) = 0$.

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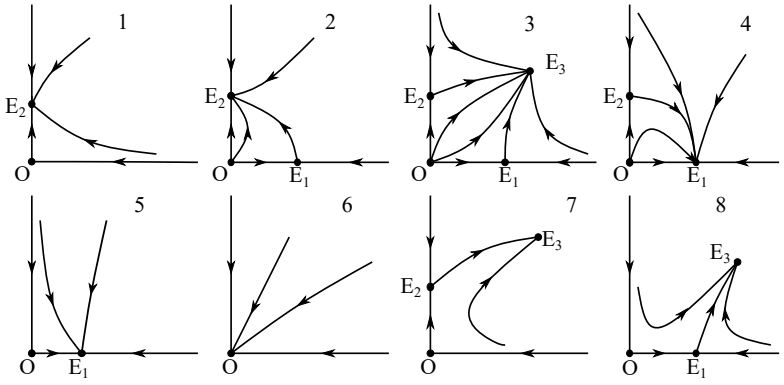


Figure 4: Phase portraits corresponding to the eight regions $1-8$ from the bifurcation diagrams $D1-D12$.

4 Conclusions

We eliminated the restriction $\theta(\mu)\cdot\delta(\mu)\neq 0$ that appears in [9] in the study of the truncated normal form of double-Hopf bifurcation. In the three cases we considered, new bifurcation diagrams have been obtained which describe the behavior of the normal form in these cases. More other cases may arise by eliminating the condition $\theta(\mu)\cdot\delta(\mu)\neq 0$, which we want to approach in further studies.

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