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DYNAMIC ANALYSIS IN AN ECONOMIC MODEL WITH EXPONENTIAL UTILITY FUNCTION AND DELAY

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Abstract

In this paper we will analyze a mathematical model associated to an economic growth process with logistic population, exponential utility function and production function of type Cobb-Douglas. Mathematical modeling of this economic growth process leads to an optimal control problem with delay. We show that the model is described by dynamical system of differential functions equations with delay which have the steady state. This steady state exhibits the Hopf bifurcation. [1]

Keywords and phrases: *delay, Hopf bifurcation, mathematical model applied in economies*

1 Introduction

The purpose of this paper is to study the Hopf bifurcation of a economical growth model with logistic population growth and delay between investment and production (time-to-build). We consider a economical growth model with logistic population growth in which production occurs with delay while new capital is installed ([3],[4],[5]). In this economy, the consumer chooses at each moment in time the level of consumption so as to maximize the global utility on the infinite time, given by an exponential function. The mathematical model of this economic growth process leads to an optimal control problem with delay. The optimality conditions, due to the introduction of the time delay, leads to a system of functional differential equations with delay. We determine the steady state of this system and we investigate the local stability of the steady state by analyzing the corresponding transcendental characteristic equation of its linearized system. In the following, by choosing delay as a bifurcation parameter, we show that this model with a delay exhibits the Hopf bifurcation ([6]).

[1] MSC(2010): 49J35, 49K35

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2 Setup the model

We consider an economy that is inhabited by infinitely-lived households that, each household has access to a technology that transforms labor and capital into output Y by a neoclassical production function ([2]). We assume that at time t household usees capital goods produced at time $t-\tau$, therefore the output at time t is determined by the following Cobb-Douglas production function

$$Y(t) = K^{\alpha}(t-\tau)L^{1-\alpha}(t),$$

where $Y(t)$ and $K(t-\tau)$ denote aggregate output and aggregate capital stock spent producing goods and $\alpha \in (0,1)$. Considering the aggregate consumption $C(t)$, the capital accumulation equation is

$$\dot{K}(t) = K^{\alpha}(t-\tau)L^{1-\alpha}(t) - C(t) - \delta K(t-\tau),$$

where $\delta \in [0,1]$ is the rate at which capital depreciates. The function L is assumed to evolve according to the logistic law

$$\dot{L}(t) = 2L(t) - L^2(t),$$

the initial population has been normalized to one.

We denote the capital per unit of labor by

$$k = \frac{K}{L}, c(t) = \frac{C(t)}{L(t},$$

for any $L \neq 0$, we can rewrite the capital accumulation equation in intensive form:

$$\dot{k}(t) = k^{\alpha}(t-\tau) - c(t) - (2 - L(t) + \delta)k(t-\tau).$$

In this economy the consumer chooses at each moment in time the level of consumption $c(t)$ such that to maximize the global utility, given by

$$-\frac{1}{\theta}\int_0^{\infty} e^{-\rho t - \theta c(t)}dt,$$

subject to the following constrains

$$\begin{aligned}
\dot{k}(t) &= k^{\alpha}(t-\tau) - c(t) - (2 - L(t) + \delta)k(t-\tau), \\
\dot{L}(t) &= 2L(t) - L^2(t) \\
k(t) &= \varphi(t), t \in [-\tau, 0],
\end{aligned}$$

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where $0 < c(t) \leq k^{\alpha}(t-\tau)$, $k(t-\tau)$ is the productive capital at time t, and $\varphi : (-\infty, 0] \to \mathbb{R}_+$ is the initial capital function; it needs to be specified in order to identify the relevant history of the state variable. That economical problem, leads us to the following mathematical optimization problem.

Problem P. To determine (c^*, k^*, L^*) which maximizes the following functional

$$-\frac{1}{\theta}\int_0^{\infty} e^{-\rho t-\theta c(t)}dt$$

and which verifies

$$\begin{aligned}
\dot{k}(t) &= k^{\alpha}(t-\tau) - c(t) - (2 - L(t) + \delta)k(t-\tau),\\
\dot{L}(t) &= 2L(t) - L^2(t)\\
k(t) &= \varphi(t), t \in [-\tau, 0].
\end{aligned}$$

To solve this optimization problem, we apply the generalized Maximal Principle for time lagged optimal control problems (see Pontryagin et al 1962, [7]). Analogues to [1], the first order conditions of this model are obtained.

Remark 1. The optimal solution of the problem (P) is a solution of the following system of differential equations:

$$\begin{aligned}
\dot{c}(t) &= -\frac{1}{\theta}(\delta + \rho + 2 - L(t) - \alpha k^{\alpha-1}(t-\tau))\\
\dot{k}(t) &= k^{\alpha}(t-\tau) - c(t) - (2 - L(t) + \delta)k(t-\tau),\\
\dot{L}(t) &= 2L(t) - L^2(t)
\end{aligned}$$

3 Local stability analysis and Hopf bifurcation

Generally, the above system is not analytically solvable but we can state some qualitative properties of the solutions. First, we determine the steady states (c^*, k^*, L^*) of the functional differential equation system, which are determined by setting $\dot{c}(t) = \dot{k}(t) = \dot{L}(t) = 0$. From system, results that we have the following

Proposition 3.1. *(Stationary state)* The system of functional differential equation has a unique steady state (c^*, k^*, L^*) which is determined by the following equations:

$$k^* = \left(\frac{\delta+\rho}{\alpha}\right)^{\frac{1}{\alpha-1}},\ c^* = \left(\frac{\delta+\rho}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \delta\left(\frac{\delta+\rho}{\alpha}\right)^{\frac{1}{\alpha-1}},\ L^* = 2.$$

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With respect to the transformation

$$x_1(t) = c(t) - c^*,\ x_2(t) = k(t) - k^*,\ x_3(t) = L(t) - L^*$$

the system becomes

$$\begin{cases} \dot{x_1}(t) = -\frac{1}{\theta}[\delta + \rho + 2 - L^* - x_3(t) - \alpha(x_2(t-\tau) + k^*)^{\alpha-1}] \\ \dot{x_2}(t) = -x_1(t) - c^* + (x_2(t-\tau) + k^*)^{\alpha} - (2 - L^* - x_3(t) + \delta)(x_2(t-\tau) + k^*) \\ \dot{x_3}(t) = x_3(t) + L^* - (x_3(t) + L^*)^2 \end{cases}$$

Expanding in Taylor series around $(0,0,0)^t$ and neglect the terms of higher order than the third order, we can rewrite system in the form

$$\begin{cases} \dot{x_1}(t) = a_{010}x_2(t-\tau) + a_{001}x_3(t) + \frac{1}{2}a_{020}x_2^2(t-\tau) + \frac{1}{6}a_{030}x_2^3(t-\tau) + ... \\ \dot{x_2}(t) = b_{100}x_1(t) + b_{010}x_2(t-\tau) + b_{001}x_3(t) + \frac{1}{2}[b_{020}x_2^2(t-\tau) + 2b_{011}x_2(t-\tau)x_3(t)] + \\ \frac{1}{6}b_{030}x_2^3(t-\tau) + ... \\ \dot{x_3}(t) = c_{001}x_3(t) + \frac{1}{2}c_{002}x_3^2(t) + ... \end{cases}$$

where
$a_{010} = \frac{\alpha(\alpha-1)}{\theta}\left(\frac{\delta+\rho}{\alpha}\right)^{\frac{\alpha-2}{\alpha-1}}, a_{020} = \frac{\alpha(\alpha-1)(\alpha-2)}{\theta}\left(\frac{\delta+\rho}{\alpha}\right)^{\frac{\alpha-3}{\alpha-1}}, a_{001} = \frac{1}{\theta},$
$a_{030} = \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{\theta}\left(\frac{\delta+\rho}{\alpha}\right)^{\frac{\alpha-4}{\alpha-1}}, b_{001} = \left(\frac{\delta+\rho}{\alpha}\right)^{\frac{1}{\alpha-1}},$
$b_{020} = \alpha(\alpha-1)\left(\frac{\delta+\rho}{\alpha}\right)^{\frac{\alpha-2}{\alpha-1}}, b_{030} = \alpha(\alpha-1)(\alpha-2)\left(\frac{\delta+\rho}{\alpha}\right)^{\frac{\alpha-3}{\alpha-1}}, b_{010} = \rho,$
$b_{011} = 1, b_{100} = -1, c_{001} = -2, c_{002} = -2.$

To investigate the local stability of steady state we linearize the last system. Letting $u(t) = (u_1(t), u_2(t))^t$, be the linearized system variables, linearized system is given by

$$\dot{u}(t) = Au(t) + Bu(t-\tau),$$

where

$$A = \begin{pmatrix} 0 & 0 & a_{001} \\ -1 & 0 & b_{001} \\ 0 & 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a_{010} & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The associated characteristic equation of the linearized system is given by:

$$\lambda^3 + 2\lambda^2 - [\rho\lambda^2 + (\rho - a_{010})\lambda - a_{010}]e^{-\lambda\tau} = 0.$$

Proposition 3.2 If $\tau = 0$ then the characteristic equation is given by

$$\lambda^3 + (2-\rho)\lambda^2 + (\rho - a_{010})\lambda + a_{010} = 0$$

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This equation has one positive eigenvalue and two eigenvalues with negative real parts.

Proposition 3.3 Let $\lambda = \lambda(\tau)$ be a solution of characteristic equation. If τ_c, ω_c are given by

$$\omega_c = \frac{\sqrt{2}}{2}\sqrt{\rho^2 + \sqrt{\rho^4 + 4a_{010}^2}},$$

$$\tau_c = \frac{1}{\omega}\arctan\frac{\rho\theta\omega}{\alpha(1-\alpha)\left(\frac{\delta+\rho}{\alpha}\right)^{\frac{\alpha-2}{\alpha-1}}}$$

and $Re\left(\frac{d\lambda}{d\tau}\right)_{\lambda=i\omega,\tau=\tau_c} \neq 0$ then a Hopf bifurcation occurs at the steady state given by (c^*, k^*, L^*) as τ passes through τ_c.

4 Conclusion

In this paper, we analyzed a growth economical model in which the utility is given by an exponential function and Cobb-Douglas production function, with delay for capital and with logistic population growth. Using the delay as a bifurcation parameter we have shown that a Hopf bifurcation occurs when this parameter passes through a critical value .

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