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Estimation of Chirp Signals by Extended Kalman Filtering

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Abstract – The paper addresses the problem of estimating the parameters of chirp signals embedded in Gaussian noise. We consider an estimation method based on an approximate linear state space representation of the polynomial phase signal. This approach offers the opportunity to use a nonlinear but exact measurement equation and guides the estimation of the states of these signals to an extended Kalman filtering algorithm. Procedure simulations were made on linear and quadratic phase modulation signals with time-varying amplitude and are consistent with the theoretical approach. The results given by this new algorithm are compared with the performances of a standard Kalman technique.

I. INTRODUCTION

Polynomial phase signals (PPS) are frequently encountered in many signal processing applications such as in radar, sonar, laser velocimetry or telecommunications. There are non-stationary signals having a fast-varying instantaneous frequency. The estimation of the parameters of PPS signals affected by additive Gaussian noise has received considerable interest in signal processing literature and several methods formulated as linear system identification problems, have been used to solve the problem [1]. These approaches admit the solution in the form of a linear Kalman filter [2]-[4], which is the optimal tracking algorithm when the signal models are assumed linear and both state and observation noise are additive and Gaussian. A linear state model can be obtained by the approximation of Tretter [3], which regards as uncorrelated both amplitude and phase components of the gaussian noise.

As the Tretter linear state model works satisfactorily as far as the signal-to-noise ratio (S/N ratio) exceeds 13dB, at lower levels of S/N ratios will be used nonlinear state models and Extended Kalman Filtering (EKF) procedures [5]-[7] which considers a local linearization that uses a first order Taylor expansion of nonlinear equations.

In this paper we consider the estimation of parameters of a variable amplitude linear chirp signal, which is a second order polynomial phase signal, corrupted by additive Gaussian noise. As compared to previous works on the subject [5], [6], the EKF algorithm developped in this paper removes their phase uncertainties by replacing the real-valued signal by its analytic representation.

A drawback of EKF algorithm are the important number of divergence cases that arises even at large S/N ratios. To overcome this limitation, the EKF algorithm that we present uses a procedure that overestimates adaptively the variance of noise in order to compensate the effect of high-order terms neglected by linearization.

This paper is organized as follows. Section 2 introduces the state-space model of variable amplitude polynomial phase signal affected by additive Gaussian noise. In section 3 we describe the EKF algorithm used in the estimation of PPS parameters. Section 4 provides simulation results and comparison with respect to linear Kalman filtering algorithm introduced in [4]. The results are obvious: in comparison with the previous algorithm, EKF works satisfactorily well at very low S/N ratios, especially with regard to polynomial phase parameters estimation Finally, section 5 gives the concluding remarks and sketches the prospective work to be done.

II. NON LINEAR STATE-SPACE REPRESENTATION OF POLYNOMIAL PHASE SIGNALS

A polynomial phase complex signal with variable amplitude embedded in additive noise is expressed as

$$y[n] = A[n]\exp\left(j\Phi[n]\right) + w[n] = A[n]\exp\left(j\sum_{i=0}^{M} b_i n^i\right) + w[n] \quad (1)$$

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where the positive real-valued $A[n]$ is the amplitude of the signal which can be constant or time varying and $\Phi[n]$ is a deterministic phase polynomial of order M, with the phase coefficients $b_i, i=0,\ldots,M$ assumed real and unknown. The additive noise is assumed complex, white and Gaussian, having zero-mean and variance σ_w^2. It can be written as

$$w[n]=w_R[n]+jw_I[n] \quad (2)$$

with $w_R[n]$ and $w_I[n]$ the real and the imaginary part of the analytical noise. If both parts are not correlated between them, having the same variance, we can write:

$$E\{w_R[n]w_R[n+k]\}=\frac{\sigma_w^2}{2}\delta[k] \quad (3)$$

$$E\{w_I[n]w_I[n+k]\}=\frac{\sigma_w^2}{2}\delta[k] \quad (4)$$

$$E\{w_R[n]w_I[n+k]\}=0, \quad \forall k\in Z \quad (5)$$

where $E\{\cdot\}$ is the expectation operator. An analytical signal having these properties is called "cyclic" noise [8].

The State-Space Model and Transition Equation

The state-space model and the transition equation of a polynomial phase signal can be derived taking as a starting point the M-order phase polynomial $\Phi[n]$ Taylor series expansion [4], [8]:

$$\Phi[n+1]=\sum_{k=0}^{M}\frac{1}{k!}\Phi^{(k)}[n] \quad (6)$$

$$\Phi^{(l)}[n+1]=\sum_{k=l}^{M}\frac{1}{(k-l)!}\Phi^{(k)}[n] \quad l=\overline{1,M} \quad (7)$$

where $\Phi^{(k)}[n]$ stands for the k-order derivative of the phase function:

$$\Phi^{(k)}[n]=\Phi^{(k-1)}[n]-\Phi^{(k-1)}[n-1], \quad k=\overline{1,M} \quad (8)$$

Note that in discrete time other definitions for (8) are possible as well [1].

In order to obtain the exact state-representation of a variable amplitude PPS, we define the following $(M+2)\times 1$ state vector $\mathbf{x}[n]$:

$$\mathbf{x}[n]=\begin{bmatrix}A[n] & \Phi[n] & \Phi^{(1)}[n] & \Phi^{(2)}[n] & \ldots & \Phi^{(M)}[n]\end{bmatrix}^T \quad (9)$$

Considering only phase variations of a PPS signal, the state transition equation is written as

$$\mathbf{x}[n+1]=\mathbf{F}\mathbf{x}[n] \quad (10)$$

where the $(M+2)\times(M+2)$ transition matrix $\mathbf{F}$ is composed of coefficients in (6) and (7)

$$\mathbf{F}=\begin{bmatrix}1 & 0 & 0 & 0 & \ldots 0\\ 0 & 1 & 1/1! & 1/2! & \ldots 1/M!\\ 0 & 0 & 1 & 1/1! & \ldots 1/(M-1)!\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \ldots 1\end{bmatrix} \quad (11)$$

This model can be extended in order to include variable amplitude PPS. We will assume that the amplitude of the signal follows a random walk model

$$A[n+1]=A[n]+v[n] \quad (12)$$

where $v[n]$ is a sequence of i.i.d. random scalars with the distribution $N(0,\sigma_v^2)$. Thus, the rate of evolution of the chirp amplitude is described by σ_v^2.

Including eq. (12) in eq. (10), the final expression of state transition equation is

$$\mathbf{x}[n+1]=\mathbf{F}\mathbf{x}[n]+\mathbf{G}v[n] \quad (13)$$

where $\mathbf{G}$ is a $(M+2)\times 1$ vector

$$\mathbf{G}=\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T \quad (14)$$

As reveals (13) the state transition equation of PPS model is linear.

The Observation Equation

In order to estimate the parameters of chirp signals corrupted by noise, a nonlinear is used. In this sense, the measured signal $y[n]$ is expressed as a 2×1 vector in terms of its real and imaginary parts:

$$\mathbf{y}[n]=\begin{bmatrix}\mathrm{Re}(\mathbf{y}[n]) & \mathrm{Im}(\mathbf{y}[n])\end{bmatrix}^T \quad (15)$$

Viewing (15), the observation equation is nonlinear:

$$\mathbf{y}[n]=\mathbf{h}(\mathbf{x}[n])+\mathbf{w}[n] \quad (16)$$

where the 2×1 nonlinear function $\mathbf{h}(\mathbf{x}[n])$ is written as

$$\mathbf{h}(\mathbf{x}[n])=\begin{bmatrix}h_1[n]\\ h_2[n]\end{bmatrix}=\begin{bmatrix}x_1[n]\cos(x_2[n])\\ x_1[n]\sin(x_2[n])\end{bmatrix} \quad (17)$$

The observation noise vector $\mathbf{w}[n]=\begin{bmatrix}w_R[n] & w_I[n]\end{bmatrix}^T$ is defined by (2)-(5). The correlation matrix is also given from the same equations

$$\mathbf{Q}_w[n]=\frac{\sigma_w^2}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad (18)$$

In order to use EKF, we apply the first order linearization procedure to $\mathbf{h}(\mathbf{x}[n])$ in (17) around the estimation of the state vector $\hat{\mathbf{x}}[n|n-1]$:

$$\mathbf{h}(\mathbf{x}[n])=\mathbf{h}(\hat{\mathbf{x}}[n|n-1])+\left.\frac{\delta\mathbf{h}}{\delta\mathbf{x}}\right|_{\mathbf{x}=\hat{\mathbf{x}}[n|n-1]}(\mathbf{x}[n]-\hat{\mathbf{x}}[n|n-1]) \quad (19)$$

with

$$\mathbf{H}[n]=\left.\frac{\delta\mathbf{h}}{\delta\mathbf{x}}\right|_{\mathbf{x}=\hat{\mathbf{x}}[n|n-1]} =\begin{bmatrix}\cos(\hat{x}_2[n|n-1]) & -\hat{x}_1[n|n-1]\sin(\hat{x}_2[n|n-1]) & 0 & \ldots & 0\\ \sin(\hat{x}_2[n|n-1]) & \hat{x}_1[n|n-1]\cos(\hat{x}_2[n|n-1]) & 0 & \ldots & 0\end{bmatrix} \quad (20)$$

As is obvious, the replacement of $\mathbf{h}(\mathbf{x}[n])$ by its first order approximation has dramatic effects on stability and convergence of EKF agorithm, which implies the appearance, especially at low S/N ratios, of "lack of convergence" cases. A mechanism which practically eliminates these cases will be presented in the next Section.

III. THE EKF ALGORITHM

As far as the observation model is nonlinear, in order to apply the Kalman filtering procedure as it was shown, a first order linearization around $\hat{\mathbf{x}}\left[n|n-1\right]$ is needed at each step of the standard Kalman algorithm. The procedure is well known as Extended Kalman Filter (EKF) algorithm [9] and it uses state-space equations (13) and (16) as well as the linearization of the observation function around the current vector estimate (20).

Assume that the initial state $\mathbf{x}[1]$, the observation noise $\mathbf{w}[n]$ and the state noise $v[n]$ are jointly Gaussian and mutually independent. Let $\hat{\mathbf{x}}\left[n|n-1\right]$ and $\mathbf{R}\left[n|n-1\right]$ be the conditional mean and the conditional variance of $\hat{\mathbf{x}}[n]$ given the observations $\mathbf{y}[1],\dots,\mathbf{y}[n-1]$ and let $\hat{\mathbf{x}}\left[n|n\right]$ and $\mathbf{R}\left[n|n\right]$ be the conditional mean and conditional variance of $\hat{\mathbf{x}}[n]$ given the observations $\mathbf{y}[1],\dots,\mathbf{y}[n]$. Then [9]

Measurement Update

$$\mathbf{H}[n]=\begin{bmatrix}\cos\left(\hat{x}_2\left[n|n-1\right]\right) & -\hat{x}_1\left[n|n-1\right]\sin\left(\hat{x}_2\left[n|n-1\right]\right) & 0 & \dots & 0\\ \sin\left(\hat{x}_2\left[n|n-1\right]\right) & \hat{x}_1\left[n|n-1\right]\cos\left(\hat{x}_2\left[n|n-1\right]\right) & 0 & \dots & 0\end{bmatrix} \quad (21)$$

$$\mathbf{K}[n]=\mathbf{R}\left[n|n-1\right]\mathbf{H}^T[n]\left(\mathbf{H}[n]\mathbf{R}\left[n|n-1\right]\mathbf{H}^T[n]+\hat{\mathbf{Q}}_w[n]\right)^{-1} \quad (22)$$

$$\hat{\mathbf{x}}\left[n|n\right]=\hat{\mathbf{x}}\left[n|n-1\right]+\mathbf{K}[n]\left(\mathbf{y}[n]-\begin{bmatrix}\hat{x}_1\left[n|n-1\right]\cos\left(\hat{x}_2\left[n|n-1\right]\right)\\ \hat{x}_1\left[n|n-1\right]\sin\left(\hat{x}_2\left[n|n-1\right]\right)\end{bmatrix}\right) \quad (23)$$

$$\mathbf{R}\left[n|n\right]=\mathbf{R}\left[n|n-1\right]-\mathbf{K}[n]\mathbf{H}[n]\mathbf{R}\left[n|n-1\right] \quad (24)$$

Time Update

$$\hat{\mathbf{x}}\left[n+1|n\right]=\mathbf{F}\hat{\mathbf{x}}\left[n|n\right] \quad (25)$$

$$\mathbf{R}\left[n+1|n\right]=\mathbf{F}\mathbf{R}\left[n|n\right]\mathbf{F}^T+\mathbf{G}\mathbf{G}^T\sigma_v^2 \quad (26)$$

where $\mathbf{K}[n]$ is the Kalman gain matrix at moment n.

The parameters of variable amplitude PPS, given by the vector $\boldsymbol{\theta}=\left[A[n]\ \ b_0\ \ b_1\ \ \dots\ \ b_M\right]^T$ can be estimated from the estimates of the state vector using the relation [5]

$$\hat{\boldsymbol{\theta}}[n]=\mathbf{C}\mathbf{F}^{-n}\hat{\mathbf{x}}\left[n|n\right] \quad (27)$$

where the matrix $\mathbf{C}$ is a diagonal:

$$\mathbf{C}=\mathrm{diag}\left(\left[1\ \ 1\ \ 1/1!\ \ \dots\ \ 1/M!\right]\right) \quad (28)$$

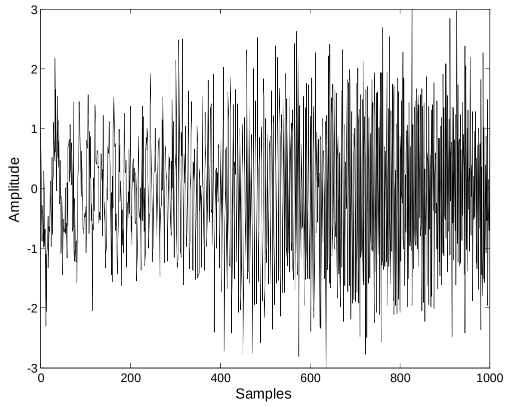

Figure 2. Second order PPS in Gaussian noise, SNR=5dB.

As EKF is not an optimal estimator, if the initial estimation of the state is wrong or if the process is modeled incorrectly, the filter may quickly diverge, owing to its linearization. This behaviour appears in our case since the S/N ratio is lower than 10dB. As example for $SNR=0\text{dB}$, the rate of divergent cases may exceed 20%. By contrast, there are not cases of divergence for the linear counterpart of the method [4]. Figure 1 shows the rate of divergence of standard EKF method with respect to SNR obtained from simulations made on a typical PPS.

As a result of many simulations carried out, we concluded that is more interesting to overestimate the value of the variance of the noise in order to compensate for the terms neglected during the linearization of measurement equation. The consequences of such an increase on variance are positive: the rate of divergence diminish drastically. The same effect is seen with respect to estimation errors. The overestimation procedure was established empirically and lies in substitution of matrix $\hat{\mathbf{Q}}_w[n]$ in (22) by $k_R\mathbf{Q}_w[n]$ where the robustness factor k_R is computed as follows

$$10\lg k_R[\text{dB}]=\begin{cases}15, & \text{if } 10\lg\sigma_w^2\le 5\\ 15-1.5\left(10\lg\sigma_w^2-5\right), & \text{if } 5<10\lg\sigma_w^2\le 15\\ 0, & \text{if } 10\lg\sigma_w^2>15\end{cases} \quad (29)$$

The improvements obtained by using k_R are revealed in Figure 1. We designate the EKF that uses k_R factor as *robust EKF* algorithm.

IV. SIMULATION RESULTS

In this section we give some simulation results for the estimation of PPS in Gaussian noise based on robust EKF algorithm. The 1000 samples second order PPS sequence presented in Figure 2 was used. The real values of its phase parameters are: $b_0 = \pi/2$, $b_1 = 0.0785$, $b_2 = 1.309\times10^{-3}$. The state noise $v[n]$ is zero-mean Gaussian white noise with $\sigma_v^2 = 10^{-3}$.

Figures 3 to 7 give the convergence plots for PPS parameters for two levels of S/N ratio. The initial conditions were as in [5]:

$$\hat{\mathbf{x}}[1|0] = \left[1/2 \quad \pi/3 \quad 0 \quad 2\cdot10^{-3}\right]^T \text{ and}$$

$$\mathbf{R}[1|0] = \text{diag}\left[k_R/2 \quad \pi^2/9 \quad \pi^2/9 \quad 4.3865\cdot10^{-6}\right]$$

By contrast to Kalman-Tretter filtering algorithm introduced in [4], the EKF works satisfactorily at low levels of S/N ratio, especially if the focus of estimation is put on phase parameters. Only the amplitude estimation is strongly affected by high levels of noise. The most exact estimation is obtained for b_2, while the initial phase b_0 is the most difficult to establish, since its estimation

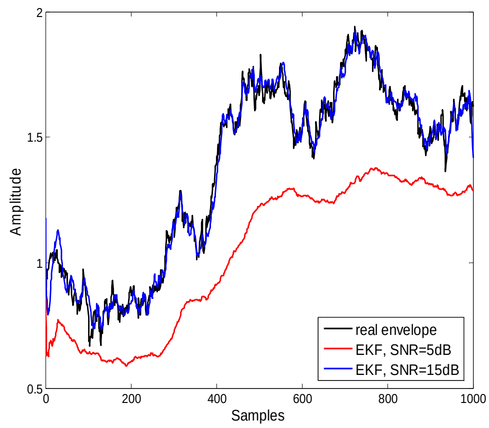

Figure 3. Amplitude estimation.

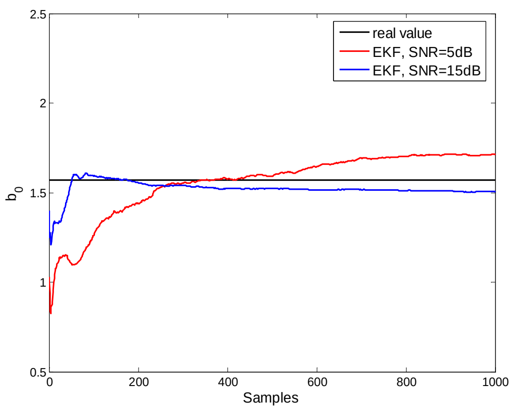

Figure 4. Estimation of b_0.

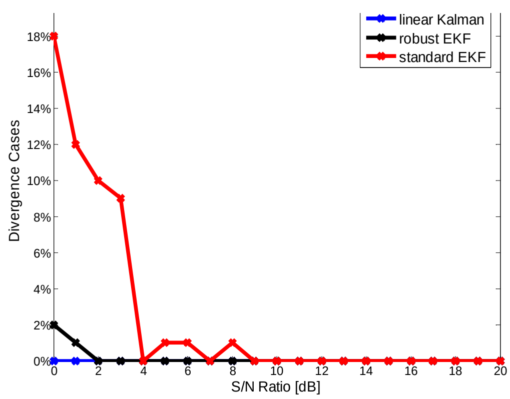

Figure 1. Rate of divergence vs. SNR assessed on the same signal by Kalman filters under disscussion.

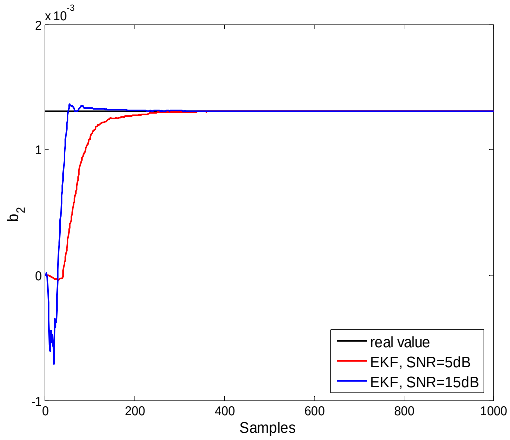

Figure 6 Estimation of b_2.

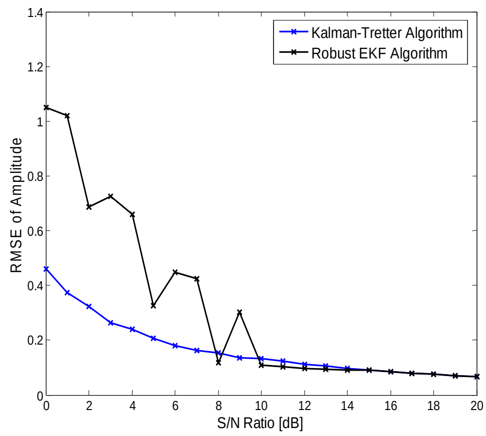

Figure 7. RMSE of amplitude vs. SNR.

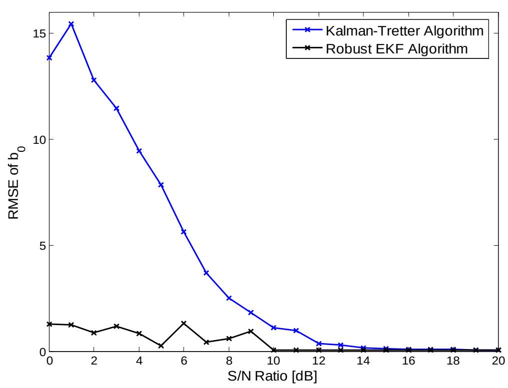

Figure 8. RMSE of b_0 vs. SNR.

depends on higher order coefficients exact estimation.

The comparison of EKF and Kalman-Tretter algorithms performances was the second goal of this paper. With that end in view, a statistical analysis was made by taking 100 noisy realizations of the test signal for S/N ratio values between 0 and 20dB. The averages of RMS error were calculated for each of 4 parameters that describe the second order PPS.

The results are presented in Figures 7 to 10 and certifies that as long the S/N ratio is lower than 13dB, the phase parameters estimation by EKF is far better

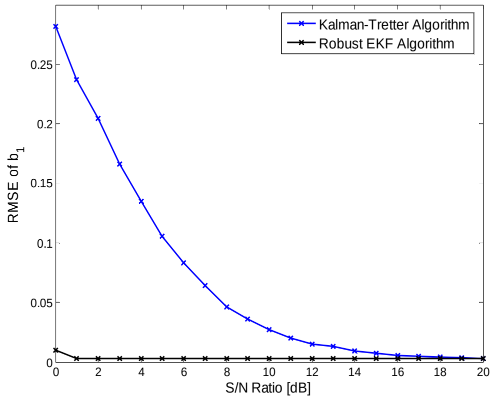

Figure 9. RMSE of b_1 vs. SNR.

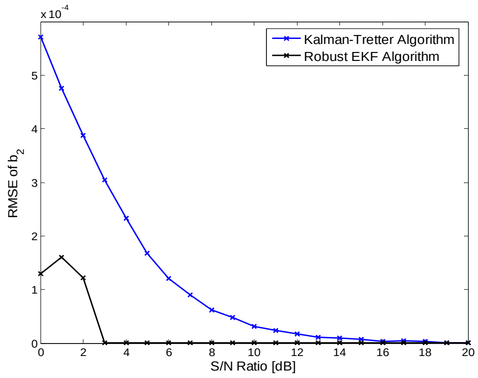

Figure 10. RMSE of b_2 vs. SNR.

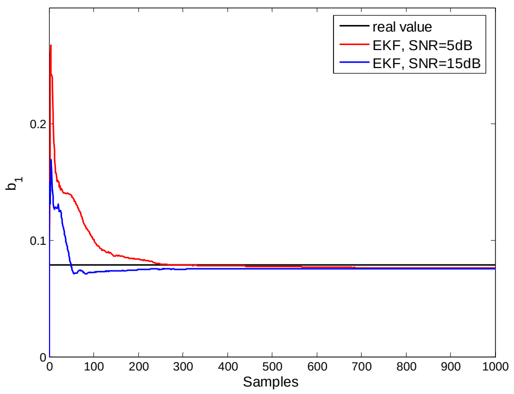

Figure 5 Estimation of b_1.

than the results given by Kalman-Tretter method. The single parameters for which Kalman-Tretter presents better performances is the amplitude, but from 10dB up, both methods performs identically. As result, we can declare that EKF extends the Kalman methods range from 13dB as imposed by linear Kalman algorithm to approximately 5dB.

The paper gives a new state space model of variable amplitude polynomial phase signals that allows better performances for EKF algorithm than the old linear Kalman method. The robust EKF implemented on this model extends the range of performances of Kalman algorithms in the polynomial phase estimation from a S/N ratio of 13dB to 5dB**.**

If the paper reveals the progress realized on the way of Kalman filtering estimation of PPS parameters, a lot of problems remain to be solved by future works. First at all, we refer to a better amplitude estimation for PPS, then to the extension to multicomponent chirp signals and higher order polynomial phase signals.

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