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Comparison of Successive Over Relaxation method and Gauss-Seidel iteration method in Steady-state groundwater flow

Beilicci Robert Florin [1] Beilicci Erika[1]

Abstract – This application solve numerically a steady-state groundwater problem in which the aim is to calculate steady-state H distribution along a two-dimensional hill slope. A clear improvement in efficiency of iterative methods is obtained if we use the newly computed values in the Gauss-Seidel iteration. Gauss-Seidel iteration method can be further be improved by increasing the convergence rate using the method of SOR (Successive Over Relaxation).

Keywords: Successive Over Relaxation, Gauss-Seidel iteration

1. INTRODUCTION

The purpose is to solve numerically a steady-state groundwater problem in which the aim is to calculate steady-state H distribution along a two-dimensional hill slope. Mathematically the hill slope is considered to be rectangular as shown in Fig. 1.

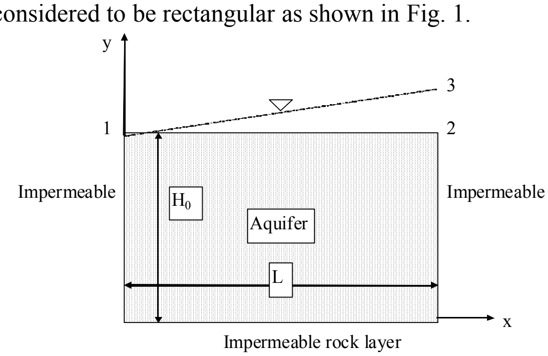

Fig. 1 Illustration of the steady state groundwater flow system.

Toth (1962) has analyzed this type of groundwater flow system and Wang and Anderson (1982) have solved the problem using FORTRAN.

Fig. 1 represents a so called deep hill slope profile bounded on the right by water divide (zero-flow boundary) and on the left by stream in the valley bottom. The left boundary is also a zero-flow boundary since there is no exchange of water between the hill slopes located on the left and right side of the stream. Aquifer bottom is assumed impermeable and therefore, it is a zero-flow boundary. Left, right and bottom boundary conditions are of Neumann type (flux specified). Water table of the aquifer is assumed to vary linearly according to line 1...3 in Fig. 1. The upper boundary of the model follows line 1...2 indicating that we need to define a Dirichlecht boundary condition as an upper boundary condition.

The aquifer is assumed to be homogenous, isotropic and in steady-state condition. The steady-state assumption is valid when average value of water table position is used as boundary condition. It can be assumed that water table position at the beginning of the year is the same as position at the end of the year, i.e. There is no net accumulation or loss of water from the system over long time periods. Therefore, the two-dimensional Laplace's equation is the required governing equation.

If hydraulic conductivity is not zero, zero-flow boundary is possible only if hydraulic gradient is zero: flux is calculated using Darcy's law,

qx = -K dH/dx = 0 if dH/dx = 0. Boundary condition at the top of the aquifer is assumed to vary linearly between point 1 and 2 in such a way that at point 1 H equals the thickness of the aquifer, H0, and increases with slope s0 and reaches its maximum at point 2 where x=L. The mathematical model of the groundwater flow system can now be summarized as:

Governing equation:

Laplace $\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}=0$

Boundary conditions:

Top $0 \le x \le L$ $H(x, y_0) = H_0 + s_0 x$

Bottom $0 \le x \le L$ $\left.\frac{\partial H}{\partial y}\right|_{Y=0}=0$

Left $0 \le y \le H_0$ $\left.\frac{\partial H}{\partial x}\right|_{X=0}=0$

Right $0 \le y \le H_0$ $\left.\frac{\partial H}{\partial x}\right|_{X=L}=0$

The first step is to substitute approximations of the second partial derivatives to Laplace's equation:

$$\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2} \approx \frac{H_{i+1,j}-2H_{i,j}+H_{i-1,j}}{(\Delta x)^2}+\frac{H_{i,j+1}-2H_{i,j}+H_{i,j-1}}{(\Delta y)^2}=0$$

If we consider a square grid in such a way that $\Delta x=\Delta y$, then simplifies to

$$H_{i,j}=(H_{i+1,j}+H_{i-1,j}+H_{i,j+1}+H_{i,j-1})/4$$

1 "Politehnica" University of Timisoara, Faculty of Civil Engineering, Department of Hydrotecnnics, Timisoara, Str. George Enescu, no. 1/A, Timisoara, Romania, www.hidro.upt.ro

In numerical solution of Laplace's equation, $H_{i,j}$ can be obtained as the average of the four neighboring nodes.

The iteration is started from the left upper corner and preceded as shown in Fig. 2.

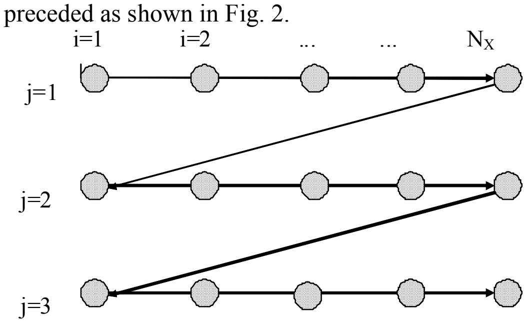

Fig. 2 Calculation order in iterative methods.

In iterative methods we need initial values at iteration level m, $H_{i,j}^{m}$ (m=0 initially) and the purpose is to calculate $H_{i,j}^{m+1}$.

Consider a grid of totally 16 nodes and hydraulic head for 12 of them is known as Dirichlecht boundary condition. There are only four unknown values in the middle of the area that needs to be calculated by numerically solving Laplace's equation.

2. GAUSS-SEIDEL ITERATION METHOD

A clear improvement in efficiency of iterative methods is obtained if we use the newly computed values in the iteration formula: iteration level m+1 values are available for nodes (i-1,j) and (i,j-1) when calculating H for node (i,j). Thus the Gauss-Seidel formula is:

$$H_{i,j}^{m+1} = (H_{i-1,j}^{m+1} + H_{i,j-1}^{m+1} + H_{i+1,j}^{m} + H_{i,j+1}^{m}) / 4$$

The numerical values of the constants of the problem are as follows: L=160 m, H0=80 m and s0= 6/160 = 0.0375. The number nodes in x-direction is NX= 11, and the corresponding value for NY=6. This implies that $\Delta x = \Delta y = 16$ m. The grid is mesh-centered indicating that nodal points are located along the boundaries (in block-centered grid the nodal points are at the center of the grid).

Dirichlecht boundary conditions are easy to apply: in those cells known H-values are directly given.

Consider that the left boundary where i=1 is a no-flow boundary. We extend the region under consideration one node left, i.e. we define a so called fictitious node where i=0. The flow across the left boundary is zero if $\partial H/ x=0$ at x=0, which implies that $(H2 - H0)/(2\Delta x) = 0$ which is possible only if $H_0 = H_2$.

In numerical solution methods Neumann-type of boundary conditions can be treated in two different ways:

1. With direct use of the fictitious nodes.
2. The iteration equations are modified for Neumann-type boundaries.

In our problem we have a no-flow boundary along the boundary where i=1.

$$H_{1,j} = (H_{2,j} + H_{0,j} + H_{1,j+1} + H_{1,j-1}) / 4$$

For left boundary this would lead to equation

$$H_{1,j} = (2H_{2,j} + H_{1,j+1} + H_{1,j-1}) / 4$$

However, it is much more elegant to use option 1 and include the fictitious nodes in the nodal network.

The solution will be obtained using EXCEL.

In EXCEL-solution this implies that for left boundary there is one extra column where a simple equation is given as shown below. Boundary nodes (1,j) are calculated in column D and the nodal value of the fictitious node C5 is replaced by value in E5 because $H_{0,j} = H_{2,j}$.

Row/Column	*C*	*D*	*E*	*F*
4	*...*	*...*	*...*	*...*
5	*=E5*	*=(D4+D6+C5+E5)/4*	*...*	*...*
6	*...*	*...*	*...*	*...*
7	*...*	*...*	*...*	*...*

The final results are shown in Table 1.

Table 1 Solution of the steady-state groundwater example using Gauss-Seidel iteration with EXCEL.

i=1	i=2	...			i=6					i=11	
80.96	81.18	81.56	82.01	82.50	83.00	83.50	83.99	84.44	84.82	85.04	j=2
81.48	81.59	81.85	82.19	82.59	83.00	83.41	83.81	84.15	84.41	84.52	j=3
81.79	81.86	82.05	82.32	82.65	83.00	83.35	83.68	83.95	84.14	84.21	j=4
81.95	82.01	82.16	82.40	82.69	83.00	83.31	83.60	83.84	83.99	84.05	j=5
82.00	82.06	82.20	82.43	82.70	83.00	83.30	83.57	83.80	83.94	84.00	j=6
81.95	82.01	82.16	82.40	82.69	83.00	83.31	83.60	83.84	83.99	84.05	Fictit. row

3. SUCCESSIVE OVER RELAXATION METHOD

Gauss-Seidel iteration method can be further be improved by increasing the convergence rate using the method of SOR (Successive Over Relaxation). The change between two successive Gauss-Seidel iterations is called the residual c, which is defined as

$$c = H_{i,j}^{m+1} - H_{i,j}^{m}$$

In the method of SOR, the Gauss-Seidel residual is multiplied by a relaxation factor ω, and new iteration value is obtained from

$$H_{i,j}^{m+1} = H_{i,j}^{m} + \omega c = H_{i,j}^{m} + \omega(\hat{H}_{i,j}^{m+1} - H_{i,j}^{m}) = (1-\omega)H_{i,j}^{m} + \omega\hat{H}_{i,j}^{m+1}$$

It can be easily seen that if ω =1, SOR reduces to Gauss-Seidel iteration method.

By substituting of the Gauss-Seidel iteration method, we obtain the equation used in the SOR method

$$H_{i,j}^{m+1} = (1-\omega)H_{i,j}^{m} + \omega(H_{i-1,j}^{m+1} + H_{i,j-1}^{m+1} + H_{i+1,j}^{m} + H_{i,j+1}^{m}) / 4$$

Usually the numerical value of relaxation parameter ω can be obtained by trial and error and optimum value is around 1.5...1.8. Remson et al. (1971) present some methods for optimum estimation of ω. In the case that 0<ω<1, the method is said to be under relaxed. According to the selection of parameter ω, we either extrapolate (ω>1) or interpolate (0<ω<1) between the old iteration value at level m and Gauss-Seidel value at level m+1. If we extrapolate too much, i.e. ω is too high; the iteration starts to oscillate and probably collapses. Therefore, in the solution of groundwater problems, ω is usually smaller than 1.8.

The changes to the program are very small, i.e. the formulas for the inner cells are replaced by the iterative equation of the SOR-method:

Row/Column	C	D	E	F	
4	...		...	...	...
5	=E5	=(1-Omega)*D5+ Omega*(D4+D6+C5+E5)/4		...	...
6	...		...	...	...
7	...		...	...	...

The solution will be obtained using EXCEL. The final results are shown in Table 2.

Table 2 Solution of the steady-state groundwater example using Successive over relaxation iteration with EXCEL

i=1	i=2	...			i=6					i=11
80.96	81.18	81.56	82.01	82.50	83.00	83.50	83.99	84.44	84.82	85.04
81.48	81.59	81.85	82.19	82.58	83.00	83.41	83.81	84.15	84.41	84.52
81.79	81.86	82.05	82.32	82.65	83.00	83.35	83.68	83.95	84.14	84.21
81.95	82.01	82.16	82.40	82.69	83.00	83.31	83.60	83.83	83.99	84.05
82.00	82.05	82.20	82.42	82.70	83.00	83.30	83.57	83.80	83.94	84.00
81.95	82.01	82.16	82.40	82.69	83.00	83.31	83.60	83.83	83.99	84.05

4. ANALYTICAL SOLUTION

Toth (1962) has published an analytical solution for this problem:

$$H(x,y)=H_0+\frac{cL}{2}-\frac{4cL}{\pi^2}\sum_{m=0}^{\infty}\frac{\cos[(2m+1)\pi x/L]\cosh[(2m+1)\pi y/L]}{(2m+1)^2\cosh[(2m+1)\pi H_0/L]}$$

The solution will be obtained using EXCEL.

Results of the comparison between the analytical solution and numerical solution using the Gauss-Seidel method is shown in Table 3.

Results of the comparison between the analytical solution and numerical solution using the SOR-method is shown in Table 4.

Table 3 Comparison of analytical and numerical solution (Gauss-Seidel method) of the steady-state groundwater flow

x	y	H(x,y)	G-S	Error (m)	Error (%)
0	80	80.008	80.000	0.008	0.009
0	64	81.026	80.960	0.066	0.081
0	48	81.522	81.483	0.038	0.047
0	32	81.816	81.788	0.027	0.034
0	16	81.975	81.952	0.023	0.028
0	0	82.026	82.004	0.022	0.027
160	0	83.974	83.996	-0.022	-0.026

Table 4 Comparison of analytical and numerical solution (SOR-method) of the steady-state groundwater flow

x	y	H(x,y)	SOR	Error (m)	Error (%)
0	80	80.008	80.000	0.008	0.009
0	64	81.026	80.960	0.066	0.081
0	48	81.522	81.483	0.038	0.047
0	32	81.816	81.788	0.027	0.034
0	16	81.975	81.952	0.023	0.028
0	0	82.026	82.004	0.022	0.027
160	0	83.974	83.996	-0.022	-0.026

V. COMPARISON OF RESULTS OBTAINED BY USING GAUSS-SEIDEL AND SOR - METHODS

Solution of the groundwater flow system of Fig. 1 can be carried out using some programming language and in this case it is necessary to do the iteration in the program. The following example uses Pascal for solving the problem with the SOR-method.

The example program includes some comments, but it necessary to point out that the no-flow boundary conditions need to update at the beginning of each iteration. Moreover, in the sweeping cycle, the old iteration value from level m must be temporarily stored to variable OldH so that it is possible to calculate the change between successive iterations. In the program the new iteration value is immediately updated and therefore it is necessary to store only one H-matrix in the computer program.

The results of the computer program are shown in Table 5 for two values of the relaxation parameter ω: Gauss-Seidel results for ω =1.0 and results for SOR-method when ω =1.7. The total number of iterations needed in the SOR-method is 39 compared with 118 needed in the Gauss-Seidel-method.

Table 5 Output of the steady-state groundwater flow system. Comparison of results obtained by using Gauss-Seidel-and and SOR-methods.

Gauss-Seidel method (ω =1.0)

```
Solution:
Number of iterations =     118   Omega=  1.00   Max.error=  9.94581e-04
  80.00  80.60  81.20  81.80  82.40  83.00  83.60  84.20  84.80  85.40  86.00
  80.95  81.17  81.55  82.01  82.49  82.99  83.49  83.98  84.43  84.82  85.03
  81.47  81.58  81.84  82.18  82.57  82.99  83.40  83.80  84.14  84.40  84.50
  81.77  81.84  82.03  82.31  82.63  82.98  83.33  83.66  83.93  84.12  84.20
  81.93  81.99  82.14  82.38  82.67  82.98  83.29  83.58  83.82  83.97  84.03
  81.98  82.03  82.18  82.40  82.68  82.98  83.28  83.55  83.78  83.92  83.98
```

SOR method (ω =1.7)

```
Solution:
Number of iterations =      39   Omega=  1.70   Max.error=  8.51289e-04
  80.00  80.60  81.20  81.80  82.40  83.00  83.60  84.20  84.80  85.40  86.00
  80.96  81.18  81.56  82.01  82.50  83.00  83.50  83.99  84.44  84.82  85.04
  81.48  81.59  81.85  82.19  82.58  83.00  83.41  83.81  84.15  84.41  84.51
  81.78  81.86  82.05  82.32  82.65  83.00  83.35  83.68  83.95  84.14  84.21
  81.95  82.00  82.16  82.40  82.68  83.00  83.31  83.60  83.83  83.99  84.05
  82.00  82.05  82.20  82.42  82.70  83.00  83.30  83.57  83.79  83.94  83.99
```

The total number of iterations needed in the SOR-method is heavily dependent on the relaxation parameter ω as indicated by the results of Table 6. In this case there is an optimum value for ωopt . If too high value for relaxation parameter is used (ω =1.9 in our example), the convergence is not attained at all, which implies that extrapolating too much can lead to instability.

The Pascal-program is given in Fig. 3.

```
Const
 NX=11; NY=6; {Number of nodes in x- and y-directions}
 dx=16.0;  SlopeGW=0.0375;  H0=80.0;
 Omega=1.7;  MaxIter=1000;  IterationStoppingCriteria=0.001;
Var
 H:array[0..Nx+1,0..Ny+1] of real; {fictitious nodes: i=0; i=NX+1: j=NY+1}
 i,j,iter:integer;
 OldH,Error,MaxErr:real;
begin                  {MAIN PROGRAM BEGINS..}
 for i:= 0 to NX+1 do    {As initial condition for all nodes H[i,j]=H0}
  for j:=0 to NY+1 do
   H[i,j]:=H0;
 for i:=1 to NX do         {Boundary condition at the top of aquifer}
  H[i,1]:=H0 + SlopeGW * DX * (i-1);
iter:=0;                              {iteration begins}
repeat
 inc(iter);
 MaxErr:=0.0;
 for j:= 1 to NY do {Left and right boundary condition at the beginning of each iteration}
  begin
   H[0,j]:=H[2,j];
   H[NX+1,j]:=H[NX-1,j];
  end;
  for i:=1 to NX do                 {Bottom boundary condition}
   H[i,NY+1]:=H[i,NY-1];
 for j:=2 to NY do                     {Sweeping node-by-node}
  begin
   for i:=1 to NX do
    begin
     OldH:=H[i,j];             {store temporarily old iteration}
     H[i,j]:=(H[i-1,j]+H[i+1,j]+H[i,j+1]+H[i,j-1])/4;
     H[i,j]:=Omega*H[i,j] + (1.0-Omega)*OldH;
     Error:=abs(H[i,j]-OldH);
     if(Error>MaxErr)then MaxErr:=Error;
    end; {end i}
  end;{end j}
until(Iter>MaxIter) or (MaxErr<IterationStoppingCriteria);
 {Continue iteration as long as MaxErr less than IterationStoppingCriteria
  or Max. number of iterations exceeded}
writeln('Solution:');
writeln('Number of iterations = ',Iter:7,'   Omega=',Omega:6:2,'   Max.error= ',MaxErr:12);
for j:=1  to NY  do
 begin
  for i:=1 to NX do
   write(H[i,j]:7:2);
  writeln;
 end;
 write('Press ENTER to continue..'); readln;
end.
```

Fig. 3 Pascal-program for solving the steady-state groundwater flow system using the SOR-method.

Table 6 Total number of iterations needed as a function of relaxation parameter □ in solving the steady-state groundwater flow system

ω	Number of iterations
1.00	118
1.10	99
1.20	86
1.30	75
1.40	64
1.50	55
1.60	46
1.70	37
1.75	39
1.80	43
1.90	no convergence

REFERENCES

[1] *J. A., Nelder and, R., Mead,* Computer Journal, vol 7., p. 308. 1965.

[2] *R., O'Neill*, Function minimization using the simplex procedure, 1971.

[2] *G.F., Pinder and W.G., Gray*, Finite element simulation in surface and subsurface hydrology. Academic Press, New York., 1977.

[3] W.*H., Press, B.P., Flannery, S.A., Teukolsky and W.T., Vetterling,,* Numerical recipes in C. The art of scientific computing. Cambridge University Press, Cambridge, 1988.

[4] *I., Remson, G.M., Hornberger and F.J., Molz*, Numerical methods in subsurface hydrology. Wiley-Interscience, New York, 1971.

[5] *K.R., Rushton and S.C., Redshaw*, Seepage and groundwater flow. Wiley, New York., 1979.

[6] *Sun Ne-Zheng,* Inverse problems in groundwater modeling. Kluwer Academic Publishers. London, 1994.

[7] *J., Toth*, A theory of groundwater motion in small drainage basins in Central Alberta, Canada. Journal of Geophysical Research, 67(11): 4375-4387, 1962.

[8] *H.F., Wang and M.P., Anderson*, Introduction to groundwater modeling: Finite difference and finite element methods. W.H. Freeman and Company, San Fransisco, 1982.