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Expansion of the Euler Bernoulli equation

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Abstract – A new notion of joint is defined. The article is concerned with the relationship between the Euler-Bernoulli equation and equation of equilibrium at the point of elastic line tip. The Euler-Bernoulli equations should be expanded according to the requirements of the motion complexity of elastic robotic systems. This yields the difference in the structure of Euler-Bernoulli equations for each mode. Mathematical model of the actuators also comprises coupling between elasticity forces. General form of the elastic line is a direct outcome of the system motion dynamics, and must be described by six equations for position and orientation of every point on that elastic line.

Keywords: robot, modeling, elastic deformation, gear, link, coupling, dynamics, kinematics, trajectory planning.

I. INTRODUCTION

The Modeling and control of elastic robotic systems has been a challenge to researchers in the last three decades. In [1], the control of robots with elastic joints in contact with dynamic environment is considered. In [2], the feedback control was formed for the robot with flexible links (two-beam, two-joint systems) with distributed flexibility, robots with flexible links. In paper [6] a nonlinear control strategy for tip position trajectory tracking of a class of structurally flexible multi-link manipulators is developed. Authors of paper [8] derived dynamic equations of the joint angle, the vibration of the flexible arm, and the contact force. The paper [9] presents an approach to end point control of elastic manipulators based on the nonlinear predictive control theory. [11] presents method for the generation of efficient kinematics and dynamic models of flexible robots. In [13] author discusses the force control problem for flexible joint manipulators.

In paper [14] the authors extend the integral manifold approach for the control of flexible joint robot manipulators from the known parameter case to the adaptive case. The author of paper [15] designed a control law for local regulation of contact force and position vectors to desired constant vectors. In paper [16] different from conventional approaches, authors focus on the design of rigid part motion control and the selection of bandwidth of rigid subsystem. In [17] the equations are derived using Hamilton's principle, and are nonlinear integro-differential equations. Mathematical model of a mechanism with one degree of freedom (DOF), with one elastic gear was defined by Spong [12] still in 1987. Based on the same principle, elasticity of gears is introduced into the mathematical model in this paper, as in papers [27], [28], [29] also. However, when the introduction of link flexibility into the mathematical model is concerned, it is necessary to point out to some essential problems in this domain.

The pertinent literature is not only lacking the relationship between the equation of equilibrium at the point of elastic line tip, the "Lumped-mass approach" (LMA) and the Euler-Bernoulli equation, the "Euler-Bernoulli approach" (EBA), but it is accustomed to treat these two approaches in totally different ways. Namely, the EBA (used in [18]…) gives the possibility to analyze flexible line form of each mode in the course of task realization. The LMA (used in [4]…) gives the possibility to analyze the motion of the any point of each mode.

We consider that EBA and LMA, are two comparable methods addressing the same problem but from different aspects [27], [28]. Mathematical model obtained by any of the methods should satisfy elementary structure of the models of elastic mechanisms known in the literature [26].

In the previous literature [18]-[22], [25], the general solution of the motion of an elastic robotic system has been obtained by considering flexible deformations as transversal oscillations that can be determined by the method of particular integrals of D. Bernoulli.

We consider that any elastic deformation can be presented by superimposing D. Bernoulli particular solutions of the oscillatory character and stationary solution of the forced character. See papers [27], [28].

First detailed presentation of the procedure for creating reference trajectory was given in [5]. In our work we synthesized reference trajectory for robot model including elastic gears and links and the presence of environment force. The reference trajectory is calculated from the overall dynamic model, when the robot tip is tracking a desired trajectory in reference regime in the absence of disturbances.

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Elastic deformation (of flexible links and elastic gears) is a quantity which is at least partly encompassed by the reference trajectory. It is assumed that all elasticity characteristics in the system (both of stiffness and damping) are "known", at least partly and at that level can be included into the process of defining the reference motion. The reference trajectory thus defined allows the possibility of applying very simple control laws via PD local feedback loops, which ensures reliable tracking of the robotic tip considered in the space of Cartesian coordinates to the level of known elasticity parameters, too. As far as the working regime of the robot is concerned we think that all forces should participate in generating elastic deformations and that it is a crude approximation to assume that elastic effects are generated only by gravitational force, or only by the environment force as in [7], or that Coriolis and centrifugal forces can be neglected altogether that elastic deviations are so small, so that inertia matrix is not dependent on them, as assumed in [10].

In our paper we do not use “assumed modes technique” proposed by Meirovitch in [25]. In our paper we form Euler Bernoulli equation but we do not use “assumed modes technique” in contrast to our contemporaries. Elastic deformation is a consequence of the overall dynamics motion of the robotic system, in our opinion. Let us emphasize once again that in this paper we propose a mathematical model solution that includes in its root the possibility for analyzing simultaneously both present phenomena – the elasticity of gears and the flexibility of links, and the idea originated from [3], but not on the same principles. We show how the continuously present environment dynamics force affects the behavior of an elastic robot system.

Our future work should be directed on implementation elasticity of gears and the flexibility of links on any model of rigid robot and also on the model of reconfigurable rigid robot as given in [30], [31] or any other type of mechanism

Section II defines the kinematics model (types of joints). Section III defines the dynamics model of elastic robotic systems. In Subsections A we define a general form of the equation of flexible line of a complex robotic system of arbitrary configuration, using Euler-Bernoulli equations. Subsection B demonstrates the relationship between the equation of elastic line motion (Euler-Bernoulli equation) and equation of motion at any point of the elastic line. Section IV analyzes the movement dynamics of a multiple DOF elastic robotic pair with elastic gear and flexible link in the presence of the second mode and environment force. Section V gives some concluding remarks.

II. KINEMATICS

Kinematics and dynamics of a robotic system are analyzed. Since elasticity elements are introduced, it is necessary to explain in detail, first of all, the kinematics of these systems in order to have dynamic modeling as efficient as possible.

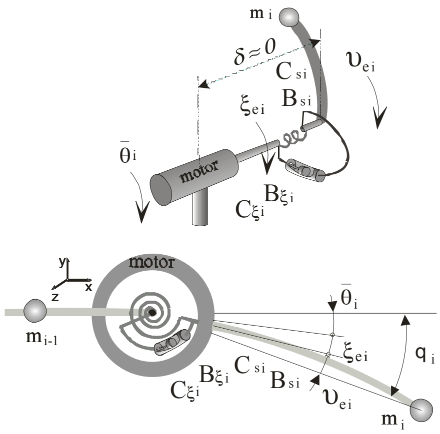

Fig.1. Spatial sketch of a rotation joint and its geometry for an elastic gear and flexible link.

Elementary type of joint is characterized by an active motor and an arbitrary elastic element (or more of them in a series) behind the motor, where elastic deformation takes place in the direction of motor motion.

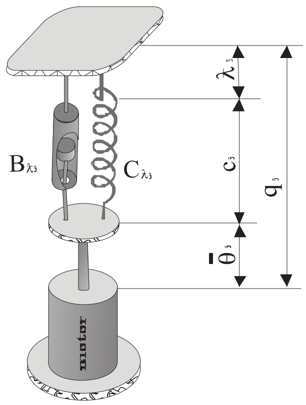

Fig. 2. Spatial sketch of a translation joint for an elastic element.

The overall coordinate q_i contains the following components: $\bar{\theta}_i$ - motor rotation angle and $\aleph_{ei}$ - elastic deformation of the elastic element behind of the motor (or $\sum\aleph_{ei}$, the sum of elastic deformations of the elastic elements in the series coming after the motor).

All these angles vary in the course of robotic task realization.

$$q_i = \bar{\theta}_i + \sum\aleph_{ei} \quad . \qquad (1)$$

This will be explained in more detail on the examples.

-Rotation joints (see Fig. 1):

The overall coordinate q_i contains the following components: $\bar{\theta}_i$, ξ_{ei} - the joint deflection angle and ϑ_{ei} - the link bending angle:

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$$q_i = \bar{\theta}_i + \xi_{ei} + \vartheta_{ei} \,. \qquad (2)$$

-Translation joints (see Fig. 2):

The angle q_i contains the following components: $\bar{\theta}_i$ and λ_i - the port deflection and also magnitude c_i that is represented length of elastic port in unstrained state.

$$q_i = \bar{\theta}_i + \lambda_{ei} + c_i \,, \quad c_i = const \,. \qquad (3)$$

It is clear that these are only special cases, when elastic deformation takes place in the direction of motor shaft deflection. Because of that it is important to define also a "kinematics" connection for the case when elastic deformation of the gear or link is not taking place in the direction of motor shaft deflection or, if there is no a drive in front of the elastic deformation (that is in front of the elastic element). By defining the type of every joint in the branched chain of the robotic system we define the configuration of that system, the transformation matrix and Jacobi matrix which is a prerequisite for defining its mathematical model and analyzing its dynamic behavior.

III. DINAMICS

A) Equation of the Elastic Line

The same interpretations of source shape of Euler-Bernoulli equation are adopted as in paper [28].

The load moment is composed of all the forces acting on the each mode of the link, and these are inertial forces (own and coupled inertia forces of the other modes), centrifugal, gravitational, Coriolis forces (own and coupled), forces due to relative motion of one mode with respect to the other, coupled elasticity forces of the other modes, as well as the force of the environment dynamics, which is via Jacobian matrix transferred to the motion of the first mode. This means that all these forces participate in generating of bending moment $\hat{\varepsilon}_{i,j} = \beta_{i,j} \cdot \dfrac{\partial^2 (\hat{y}_{i,j} + \eta_{i,j} \cdot \dot{\hat{y}}_{i,j})}{\partial \hat{x}^2_{i,j}}$ that is in forming elastic deformation as well as of the elasticity line of the each mode. All marcs are used from paper [28].

Let us consider a robotic system with m links, whereby the first link contains n_1 modes, the second link n_2 modes, etc, the m-th link contains n_m modes. On considering Fig. 3 we can see possible positions of the tip of link elastic line with n_1 modes. Model of the elastic line of this complex elastic robotic system is given in the matrix form by the following Euler-Bernoulli equation:

$$\hat{H} \cdot \frac{d^2 \hat{y}}{dt^2} + \hat{h} + j_e^T \cdot F_{uk} + z \cdot \Theta \cdot \varepsilon + \hat{\varepsilon} = 0 \,. \quad (4)$$

If we define $k = \sum_{i=1}^{m} n_i$ then we have that

$\hat{H} \in R^{kxk}$ - matrix characterizing the inertia, $\hat{h} \in R^{kx1}$ - vector of the centrifugal, gravitational and Coriolis forces, $j_e^T \in R^{kx6}$ - Jacobian matrix mapping the effect of the dynamic contact force F_{uk}, $\Theta \in R^{kxk}$ - matrix characterizing the robot configuration,

$$z = \begin{bmatrix} 0 & -\frac{1}{2^1} & \frac{1}{2^2} & -\frac{1}{2^3} & \dots & (-1)^{(k-1)} \frac{1}{2^{(k-1)}} \\ 0 & 0 & -\frac{1}{2^1} & \frac{1}{2^2} & \dots & (-1)^{(k-2)} \frac{1}{2^{(k-2)}} \\ 0 & 0 & 0 & -\frac{1}{2^1} & \dots & (-1)^{(k-3)} \frac{1}{2^{(k-3)}} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

$z \in R^{kxk}$ - characterizing matrix the mutual influence of the forces of elastic modes of all the links.

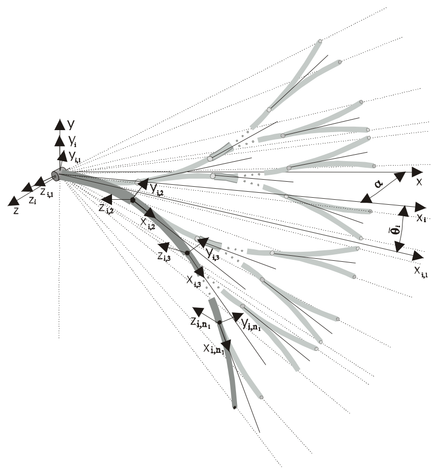

Fig. 3. Possible positions of the tip of link elastic line with n_1 modes.

$$\varepsilon = \begin{bmatrix} \varepsilon_{1,1} & \varepsilon_{1,2} \dots \varepsilon_{1,n_1} & \varepsilon_{2,1} & \varepsilon_{2,2} \dots \varepsilon_{2,n_2} & \dots\dots & \varepsilon_{m,n_m} \end{bmatrix}^T .$$

$$\hat{\varepsilon} = \begin{bmatrix} \beta_{1,1} \cdot \dfrac{\partial^2 (\hat{y}_{1,1} + \eta_{1,1} \cdot \dot{\hat{y}}_{1,1})}{\partial \hat{x}^2_{1,1}} \dots \beta_{n_m, n_{ml}} \cdot \dfrac{\partial^2 (\hat{y}_{n_m, n_{ml}} + \eta_{n_m, n_{ml}} \cdot \dot{\hat{y}}_{n_m, n_{ml}})}{\partial \hat{x}^2_{n_m, n_{ml}}} \end{bmatrix}^T .$$

Equation (4) represents the equation of motion of the elastic line of the overall robotic system.

Solution of the system (4) and dynamic motor motion, i.e. the form of its elastic line, can be obtained by superimposing the solutions for all the links involved in the presence of the dynamics (angle) of rotation of each motor, as well as by taking into account the robotic configuration, i.e. the angle α between the axes z_{i-1} and z_i.

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$$
\begin{aligned}
\hat{y} &= \hat{a}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \overline{\theta}, \alpha, t), \\
\hat{x} &= \hat{b}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \overline{\theta}, \alpha, t), \\
\hat{z} &= \hat{c}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{sti,j}, \overline{\theta}, \alpha, t), \\
\hat{\psi} &= \hat{d}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \overline{\theta}, \alpha, t), \\
\hat{\xi} &= \hat{e}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \overline{\theta}, \alpha, t), \\
\hat{\varphi} &= f(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \overline{\theta}, \alpha, t).
\end{aligned} \tag{5}
$$

Thus we defined the position and orientation of each point of the elastic line in the space of Cartesian coordinates. It should be pointed out that the form of elastic line comes out directly from the dynamics of the system motion.

B) Relationship between the Equation of elastic line motion (Euler-Bernoulli equation) and equation of motion at any point of the elastic line

The equation of motion of all the forces at the point of each mode tip of any link can be defined from (4) by setting the boundary conditions. Vector equation of all the forces involved for each mode tip of any link is:

$$
H\frac{d^2y}{dt^2} + h + j_e^T \cdot F_{uk} + z \cdot \Theta \cdot \varepsilon + \varepsilon = 0\,. \tag{6}
$$

This equation should be supplemented by the vector equation of the mathematical model of motor written in following vector form:

$$
\begin{aligned}
u &= R \cdot i + C_E \cdot \dot{\overline{\theta}} \\
C_M \cdot i &= I \cdot \ddot{\overline{\theta}} + B_u \cdot \dot{\overline{\theta}} - S \cdot (z_m \cdot \varepsilon + \varepsilon_m)
\end{aligned}\,. \tag{7}
$$

Let us define it by setting for each motor the equation of motion of all the moments acting about the rotation axis of the given motor. It has the form of the mathematical model of the motor of a rigid robotic system, but the difference being in that the moment of the i-th motor is not opposed by the mechanism moment (as with rigid robotic systems). The motor moment is opposed by the bending moment of the first elastic mode that comes after the motor, and also in part, by the bending moments of the other elastic modes that are connected in series after the given motor. All the modes after the motor, due to their position, influence the dynamics of motor motion. In (7) we have m equations of motors.

$(z_m \cdot \varepsilon + \varepsilon_m)$ is the matrix characterizing the effect of elasticity moment of each mode on the motor motion dynamic.

Matrix z_m has similar form as mentioned z matrix.

$\varepsilon_m = \begin{bmatrix} \varepsilon_{1,1} & \varepsilon_{2,1} & \varepsilon_{3,1} & \ldots\ldots & \varepsilon_{m,1} \end{bmatrix}^T$. The overall order of the system (7-8) is $(k + m)$.

The robot tip motion is defined by the sum of the stationary and oscillatory motion of each mode tip plus the dynamics of motion of the motor powering each link, as well by the included robot configuration. We can calculate the position and of each mode tip, of each link, and finally, of the robot tip motion.

Generally, we can derive the following conclusion:
To define the form of elastic line of the considered robotic system it is necessary to expand the previously known solutions, namely:
Supplement it by adding stationary solution to the particular solution of D. Bernoulli, which is of oscillatory character. This means that the given solution depends directly on the overall system dynamics. General form of the elastic line is a direct outcome of the dynamics of system motion and cannot be represented by one equation but three equations are needed to define position y, x, z and three equations ψ, ξ, φ to define orientation of each point on the elastic line. The equation of elastic line of the robotic system should also encompass the angles of motor shaft rotation $\overline{\theta}$ as in [20], the robot configuration as well, i.e. the angles between the axes z_{i-1} and z_i.
There are two aspects in defining the reference trajectory of the motor angle (see [27], [28] and [29]), viz.:
1) Elastic deformation is considered as a quantity which is not encompassed by the reference trajectory.
2) Elastic deformation is a quantity which is at least partly encompassed by the reference trajectory.

IV. EXAMPLE OF SIMULATION

Elastic deformation is a quantity which is at least partly encompassed by the reference trajectory as explained in [28] (2.1 under 2).

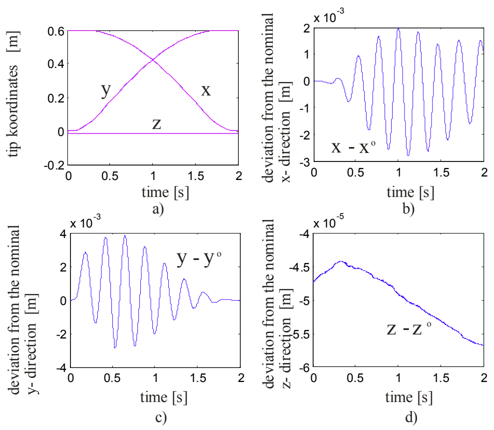

Fig. 4. Tip coordinates and deviation of position from the reference.

The characteristics of stiffness and damping of the gear in the real and reference regimes are not the same and neither are the stiffness and damping characteristics of the link.

$C_\xi = 5.0 \cdot C^o_\xi$, $B_\xi = 5.0 \cdot B^o_\xi$, $C_{s1,1} = 0.99 \cdot C^o_{s1,1}$,

$B_{s1,1} = 0.99 \cdot B^o_{s1,1}$, $C_{s1,2} = 0.99 \cdot C^o_{s1,2}$,

$B_{s1,2} = 0.99 \cdot B^o_{s1,2}$. $dt = 0.000053335(s)$.

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All other characteristics of the system and environment are the same as in paper [28]. As can be seen from Fig. 4 and Fig. 5 in its motion the robot tip tracks well the position reference trajectory and environment force, respectively in the space of Cartesian coordinates.

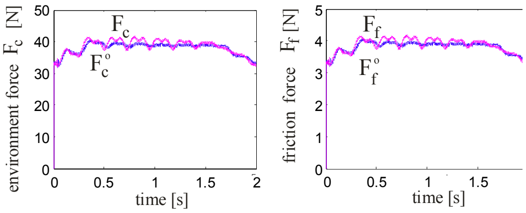
Fig. 5. The environment forces.

The elastic deformations that are taking place in the vertical plane angle of bending of the lower part of the link (the first mode) ϑ_m and the angle of bending of the upper part of the link (the second mode) ϑ_e, as well as elastic deformations taking place in the horizontal plane: the angle of bending of the lower part of the link (the first mode) ϑ_q, the angle of bending of the upper part of the link (the second mode) ϑ_δ and the deflection angle of gear ξ are given in Fig. 6.

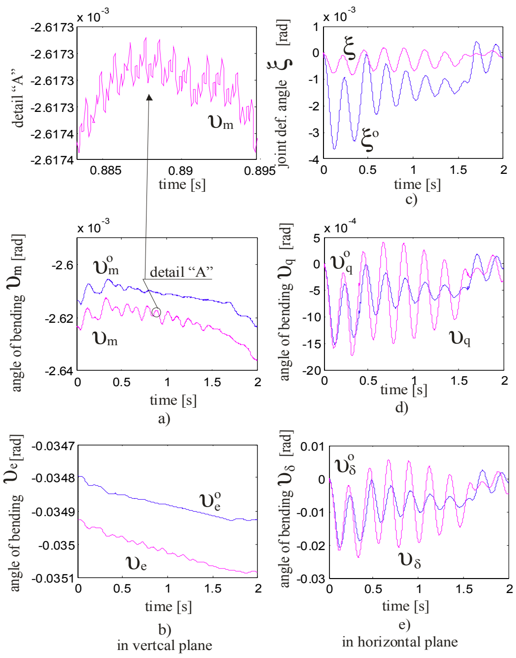
Fig. 6. The elastic deformations.

Let us show the special significance of results from Fig. 6a). These Fig. exhibit the wealth of different amplitudes and circular frequencies of the present modes of elastic elements. We have oscillations within oscillations.

This confirms that we have modeled all elastic elements as well as high harmonics (in this case two harmonics of considered link).

V. CONCLUSION

A joint is defined in a new way, in dependence of the motor state (active or locked) and type of elastic or rigid element (gear and/or link) that follows behind the motor. With so defined types of joints that may appear in a robotic construction it is possible to use the known equations to calculate the matrices of transformation and Jacobi matrix.

Based on the EBA, we defined the model of the elastic line of complex elastic robotic system with m segments, and each segment has n_i modes and also the mathematical model of motors which move each link.

We demonstrated that the equation of motion of all the forces involved at any point follows directly from the equation of elastic line. If we define boundary conditions for the mode tip as the most interesting point on the elastic line, we obtain the equation of motion at that point, what is classical form of the mathematical model of the elastic robotic system considered, which essentially LMA is. Thus we demonstrated the connection of the LMA and EBA. LMA is just a special case of EBA. In addition to the comparative analysis of the EBA and LMA, the paper also analyzes a number of other phenomena that make constitutive parts of the motion dynamics of these systems.

An analysis was made of the choice of reference trajectory, which depends on the level of knowing elasticity characteristics. The estimated elasticity characteristics may be included into the reference trajectory, and thus into the control law.

a) Euler-Bernoulli equation has been expanded from several aspects:
   1) Euler-Bernoulli equation (based on the known laws of dynamics) should be supplemented with all the forces that are participating in the formation of the bending moment of the considered mode, what causes the difference in the structure of these equations for each mode.
   2) Structure of the stiffness matrix must also have the elements outside the diagonal, because of the existence of strong coupling between the elasticity forces involved.
   3) Damping is an omnipresent elasticity characteristic of real systems, so that it is naturally included in the Euler-Bernoulli equation.
   4) General form of the transversal elastic deformation is defined by superimposing particular solutions of oscillatory character (solution of Daniel Bernoulli) and stationary solution of the forced character (which is a consequence of the forces involved).

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5) General form of the elastic line is a direct outcome of the dynamics of system motion and cannot be represented by one scalar equation but three equations are needed to define the position and three equations to define the orientation of each point on the elastic line.

b) Structure of the mathematical models of actuators: With elastic robotic systems, the actuator torque is opposed by the bending moment of the first elastic mode, which comes after the motor, and partly by the bending moments of other modes, which are connected in series after the motor considered. All modes coming after the motor, because of their position, exert influence on the dynamics of motor motion. The mathematical model in our paper is connected to the rest of the mechanism via the equivalent elasticity moment.

New structures of the matrix z and also z_m appear as a consequence of the coupling between the modes of particular links.

Elastic deformation is a consequence of the overall dynamics of the robotic system, what is essentially different from the method that was used until today, which purports usage of "assumed modes technique".

All this has been presented for a relatively simple robotic system that offered the possibility of analyzing the phenomena involved. Through the analysis and modeling of an elastic mechanism we made an attempt to give a contribution to the development of this area.

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