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A Kalman Filtering Algorithm for the Estimation of Chirp Signals in Gaussian Noise

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Abstract – The paper addresses the problem of estimating the parameters of polynomial phase signals (PPS) embedded in Gaussian noise. We consider an estimation method based on an approximate linear state space representation of the polynomial phase signal. This approach offers the opportunity to use a standard Kalman filtering procedure in view to estimate the parameters of PPS signals. Procedure simulations were made on linear chirp sinusoids with time-varying amplitude and are consistent with the theoretical approach. The paper presents the most important results.
Keywords: Kalman filter, polynomial phase signals, linear state model, parametric identification

I. INTRODUCTION

Chirp signals are frequently encountered in many signal processing applications such as in radar, sonar, laser velocimetry or telecommunications. The estimation of the parameters of chirp signals affected by additive Gaussian noise has received considerable interest in signal processing literature and several methods formulated as linear system identification problems, have been used to solve the problem [1]. These approaches admit the solution in the form of a Kalman filter [2]-[6], which is the optimal tracking algorithm when the signal models are assumed linear and both state and observation noise are additive Gaussian.

The parametric estimation by Kalman filtering has been largely investigated in the case of polynomial phase signals affected by Gaussian noise. The use of Kalman filter is justified by its practical advantages in the tracking of the frequency of a signal in several practical applications [3]. The first works in the field have been devoted to the identification of chirp signals. A linear state model can be obtained by the approximation of Tretter [4] which transforms the additive noise into a noise on the phase. This model is linear and Gaussian, allowing the application of the Kalman filter which is optimal from the view of the minimum of variance, being applicable in the case of monocomponent signals at moderate levels of additive noise [5], [6].

In this paper we consider the estimation of parameters of a chirp signal (also called second order polynomial phase signals) corrupted by additive Gaussian noise. In our approach, we consider the approximate linear state-space model derived in [6] for polynomial phase signals, but we propose a random walk assumption for the time evolution of the amplitude of chirp. This assumption adjoins the amplitude to the linear phase parameters which can be estimated by the algorithm described in the paper.

This paper is organized as follows. Section II introduces the state-space model of variable amplitude polynomial phase signal affected by additive Gaussian noise. In section III we describe the Kalman filter algorithm used in the estimation of chirp signal parameters. Section IV provides simulation results which confirm the validity of the model at moderate levels of noise. Finally, section V gives the concluding remarks and sketches the prospective work to be done.

II. STATE-SPACE REPRESENTATION OF CHIRP SIGNAL

The linear state-space model associated with a variable amplitude linear chirp signal is described by two equations: a state transition equation and an observation equation:

$$\begin{aligned} \mathbf{x}[n+1] &= \mathbf{F}\mathbf{x}[n] + \mathbf{G}\mathbf{v}[n] \\ \mathbf{y}[n] &= \mathbf{H}\mathbf{x}[n] + \mathbf{w}[n] \end{aligned} \tag{1}$$

$\mathbf{x}[n]$ is the state vector and $\mathbf{y}[n]$ is the observation vector, $\mathbf{v}[n]$ represent the state noise vector and $\mathbf{w}[n]$ is the vector of noise in the measured signal. $\mathbf{F}$ and $\mathbf{H}$ are the state transition matrix, respectively the observation matrix.

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__The Observation Model__

The model of a variable amplitude second order polynomial phase signal $y[n]$ embedded in the additive noise $w[n]$ is given below:

$$y[n]=A[n]\exp\left(j\Phi[n]\right)+w[n] \qquad (2)$$

where the positive real-valued $A[n]$ is the amplitude possibly time-varying and unknown and $\Phi[n]$ is the deterministic polynomial phase, expressed, for a linear chirp, by:

$$\Phi[n]=\frac{\alpha}{2}n^2+\beta n+\gamma \qquad (3)$$

where the coefficients α, β and γ are real and unknown. The additive white Gaussian noise $w[n]$ has zero-mean and variance σ_w^2. It can be written as

$$w[n]=w_R[n]+jw_I[n] \qquad (4)$$

with $w_R[n]$ and $w_I[n]$, the real part and the imaginary part of the analytical noise. If both parts are not correlated between them, having the same variance, we can write:

$$E\left\{w_R[n]w_R[n+k]\right\}=\frac{\sigma_w^2}{2}\delta[k] \qquad (5)$$

$$E\left\{w_I[n]w_I[n+k]\right\}=\frac{\sigma_w^2}{2}\delta[k] \qquad (6)$$

$$E\left\{w_R[n]w_I[n+k]\right\}=0,\quad \forall k\in Z \qquad (7)$$

where $E\{\cdot\}$ is the expectation operator. An analytical signal having these properties is called "cyclic" noise [7].

In order to estimate the parameters of chirp signals corrupted by noise, we use an adequate model of the signal with emphasis on its instantaneous phase. In this sense we express the measured signal $y[n]$ in terms of its polar components:

$$\mathbf{y}[n]=\left[\left|y[n]\right| \quad Arg\{y[n]\}\right]^T \qquad (8)$$

It is the observation vector of chirp linear model. As is shown in [4], if the signal-to-noise ratio (SNR) in the measured signal $y[n]$ exceeds $13\mathrm{dB}$, the noise real part affects only the amplitude $A[n]$, whereas the phase $\Phi[n]$ is affected by the imaginary part of the "cyclic" noise. Eq. (2) can be written now in terms of amplitude and phase as:

$$\begin{bmatrix}\left|y[n]\right| \\ Arg\{y[n]\}\end{bmatrix}=\begin{bmatrix}A[n] \\ \Phi[n]\end{bmatrix}+\begin{bmatrix}w_R[n] \\ w_I[n]/A[n]\end{bmatrix} \qquad (9)$$

Consequently, the noise observation vector in (1), $\mathbf{w}[n]$, is expressed as

$$\mathbf{w}[n]=\left[w_R[n] \quad w_I[n]\right]^T \qquad (10)$$

The correlation matrix of noise vector $\mathbf{Q}_w[n]$ is established under the assumptions (5)-(7) and the decomposition in (9), as

$$\mathbf{Q}_w[n]=\frac{\sigma_w^2}{2}\begin{bmatrix}1 & 0 \\ 0 & 1/A^2[n]\end{bmatrix} \qquad (11)$$

As the amplitude $A[n]$ is variable, $\mathbf{Q}_w[n]$ is recalculated for each step of the filtering algorithm.

__The State-Space Model and Transition Equations__

The values of an M-order polynomial $P(x)$, can be expressed by the Taylor series expansion [7]:

$$P(x_0+\Delta x)=\sum_{k=0}^{M}\frac{(\Delta x)^k}{k!}P^{(k)}(x_0);\forall x_0,\forall\Delta x\in R \qquad (12)$$

viewing that all derivatives having the order higher than M are zero. For the l-order derivative of the polynomial $P^{(l)}(x)$ can be used the following series expansion:

$$P^{(l)}(x_0+\Delta x)=\sum_{k=l}^{M}\frac{(\Delta x)^k}{(k-l)!}P^{(k)}(x_0);\ \forall x_0,\forall\Delta x\in R, l=\overline{1,M} \qquad (13)$$

Replacing $P(x)$ by the phase polynomial $\Phi[n]$ in(3) , x_0 with n and Δx by 1, we have for:

$$\Phi[n+1]=\sum_{k=0}^{M}\frac{1}{k!}\Phi^{(k)}[n] \qquad (14)$$

$$\Phi^{(l)}[n+1]=\sum_{k=l}^{M}\frac{1}{(k-l)!}\Phi^{(k)}[n] \quad l=\overline{1,M} \qquad (15)$$

The state vector $\mathbf{x}[n]$ in the case of variable amplitude linear chirp signals is given by the amplitude of the sinusoid, the phase and the first $M=2$ derivatives of the phase:

$$\mathbf{x}[n]=\left[A[n] \quad \Phi[n] \quad \Phi^{(')}[n] \quad \Phi^{('')}[n]\right]^T \qquad (16)$$

where:

$$\Phi^{(')}[n]=\Phi[n]-\Phi[n-1] \qquad (17)$$

and:

$$\Phi^{('')}[n]=\Phi^{(')}[n]-\Phi^{(')}[n-1] \qquad (18)$$

Note that in discrete time, other definitions for (17) and (18) are possible as well [1].

For two consecutive moments, the relation between the two states is derived from (14) and (15):

$$\begin{bmatrix}A[n+1] \\ \Phi[n+1] \\ \Phi^{(')}[n+1] \\ \Phi^{('')}[n+1]\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{1!} & \frac{1}{2!} \\ 0 & 0 & 1 & \frac{1}{1!} \\ 0 & 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}A[n] \\ \Phi[n] \\ \Phi^{(')}[n] \\ \Phi^{('')}[n]\end{bmatrix} \qquad (19)$$

It is a transition equation between two consecutives states, without taking into account the state noise. Comparing (19) with the first equation in (1), we find that the 4×4-size transition matrix is:

$$\mathbf{F}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & \frac{1}{1!} & \frac{1}{2!}\\ 0 & 0 & 1 & \frac{1}{1!}\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{20}$$

One of the most frequently used models of a time evolution of signal parameters (here the amplitude) is a random walk. In particular, it is assumed that the instantaneous amplitude of chirp has random increments having a Gaussian distribution. For this reason we consider for variable amplitude the following random walk model

$$A[n+1]=A[n]+v[n] \tag{21}$$

where $v[n]$ is a sequence of i.i.d. random scalars with the distribution $N\left(0,\sigma_v^2\right)$. Thus, the rate of evolution of the chirp amplitude is described by σ_v^2.

The last equation must be added to (19) in order to obtain the complete description of the state evolution for a variable amplitude chirp signal:

$$\mathbf{x}[n+1]=\mathbf{F}\mathbf{x}[n]+[1 \quad 0 \quad 0 \quad 0]^T v[n] \tag{22}$$

It results

$$\mathbf{G}=[1 \quad 0 \quad 0 \quad 0]^T \tag{23}$$

Finally we can rewrite (9) as

$$\mathbf{y}[n]=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\mathbf{x}[n]+\begin{bmatrix}w_R[n]\\ w_I[n]/A[n]\end{bmatrix} \tag{24}$$

which means that the observation matrix $\mathbf{H}$ is

$$\mathbf{H}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix} \tag{25}$$

III. KALMAN FILTERING ALGORITHM

The system identification problem for the state model in (1) can be solved by a standard Kalman filter.

Assume that the initial state $\mathbf{x}[1]$, the observation noise $\mathbf{w}[n]$ and the state noise $v[n]$ are jointly Gaussian and mutually independent. Let $\hat{\mathbf{x}}[n|n-1]$ and $\mathbf{R}[n|n-1]$ be the conditional mean and the conditional variance of $\hat{\mathbf{x}}[n]$ given the observations $\mathbf{y}[1],\ldots,\mathbf{y}[n-1]$ and let $\hat{\mathbf{x}}[n|n]$ and $\mathbf{R}[n|n]$ be the conditional mean and conditional variance of $\hat{\mathbf{x}}[n]$ given the observations $\mathbf{y}[1],\ldots,\mathbf{y}[n]$. Then [8]

Measurement Update

$$\mathbf{K}[n]=\mathbf{R}[n|n-1]\mathbf{H}^T\left(\mathbf{H}\mathbf{R}[n|n-1]\mathbf{H}^T+\hat{\mathbf{Q}}_w[n]\right)^{-1} \tag{26}$$

$$\hat{\mathbf{x}}[n|n]=\hat{\mathbf{x}}[n|n-1]+\mathbf{K}[n]\left(\mathbf{y}[n]-\mathbf{H}\hat{\mathbf{x}}[n|n-1]\right) \tag{27}$$

$$\mathbf{R}[n|n]=\mathbf{R}[n|n-1]-\mathbf{K}[n]\mathbf{H}\mathbf{R}[n|n-1] \tag{28}$$

Time Update

$$\hat{\mathbf{x}}[n+1|n]=\mathbf{F}\hat{\mathbf{x}}[n|n] \tag{29}$$

$$\mathbf{R}[n+1|n]=\mathbf{F}\mathbf{R}[n|n]\mathbf{F}^T+\mathbf{G}\mathbf{G}^T\sigma_v^2 \tag{30}$$

where $\mathbf{K}[n]$ is the Kalman gain matrix at moment n. Since an exact value of correlation matrix (11) is not available, it is estimated by $\hat{\mathbf{Q}}_w[n]$, computed at each step of Kalman algorithm as

$$\hat{\mathbf{Q}}_w[n]=\frac{\sigma_w^2}{2}\begin{bmatrix}1 & 0\\ 0 & 1/\hat{A}^2[n|n-1]\end{bmatrix} \tag{31}$$

In order to evaluate the parameters of variable amplitude linear chirp given by $\boldsymbol{\theta}[n]=\left[A[n] \quad \gamma \quad \beta \quad \alpha\right]^T$ one uses the following relation [2]

$$\boldsymbol{\theta}[n]=\mathbf{C}\mathbf{F}^{-n}\hat{\mathbf{x}}[n|n] \tag{32}$$

where the matrix $\mathbf{C}$ is a diagonal with elements 1, 1, 1, 0.5.

IV. EXPERIMENTAL RESULTS

In order to implement the state-space model introduced before we used Hilbert transformation followed by modulus and phase calculation to obtain the Cartesian coordinates decomposition of eq. (8). These data represent the measured input vector for a Kalman filtering algorithm based on one-step prediction, which is implemented in MATLAB.

The chirp signal used for tests, shown in Fig. 1, is 5000 samples long and the sampling frequency is 5000Hz. The chirp parameters have the following true values: $\alpha=1.0053\times10^{-4}$, $\beta=0.1256$ and $\gamma=\pi/2$. The state noise $v[n]$ is zero mean Gaussian white noise with $\sigma_v^2=1.799\times10^{-3}$.

The observation noise $w[n]$ is zero mean Gaussian white noise with $SNR=20\text{dB}$.

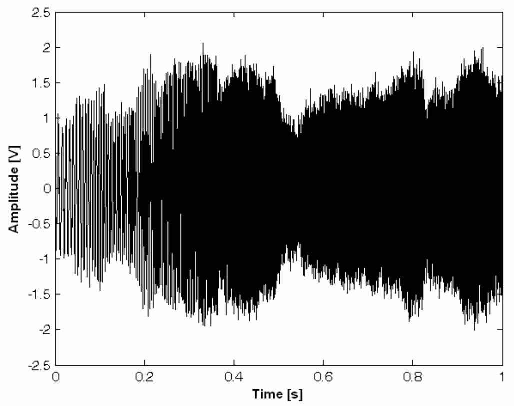

Figure 1. Chirp signal corrupted by noise used for simulation.

To give a better understanding of Kalman filter action on polynomial phase signals, Fig. 2 shows the results of instantaneous frequency estimations and Fig.3 presents the amplitude estimation obtained for the test signal.

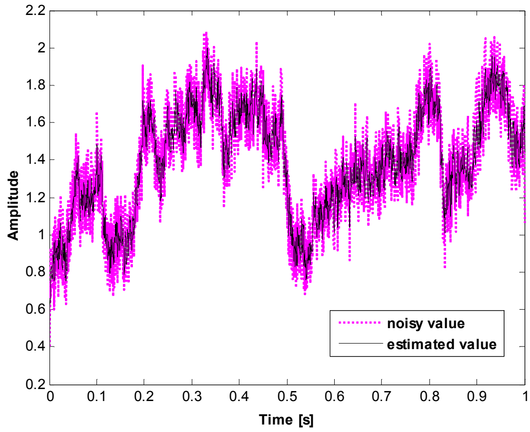

Figure 2. Estimation of the instantaneous frequency of chirp signal.

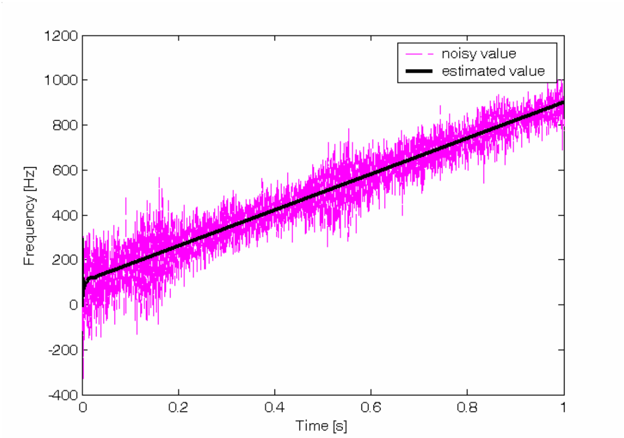

Figure 3. Amplitude estimation.

In order to evaluate the performances of Kalman filter in frequency and amplitude estimation for linear chirp signals, Fig. 4 and Fig. 5 shows how the SNR affects the RMSE of these parameters. The results certify that as long SNR exceeds 13dB, the linear model assumed for the chirp signal works well.

V. CONCLUSIONS

The paper gives the state-space model of variable amplitude polynomial phase signals with good opportunities in the estimation of parameters of linear chirp signals embedded in Gaussian white noise. The Kalman filter algorithm performs well viewing the results. The algorithm allows the extension to multicomponent chirp signals and higher order polynomial phase signals with constant as well as variable amplitude which will form the object of our future work.

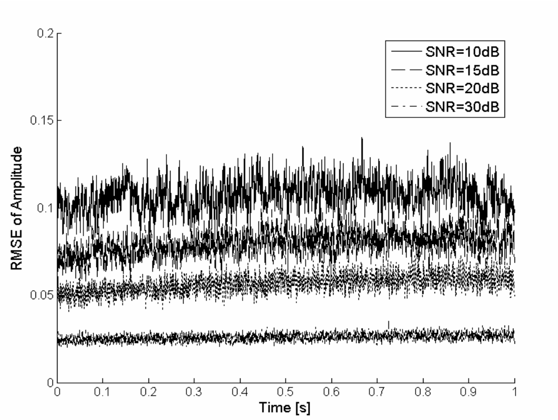

Figure 4. RMSE of estimated amplitude versus SNR.

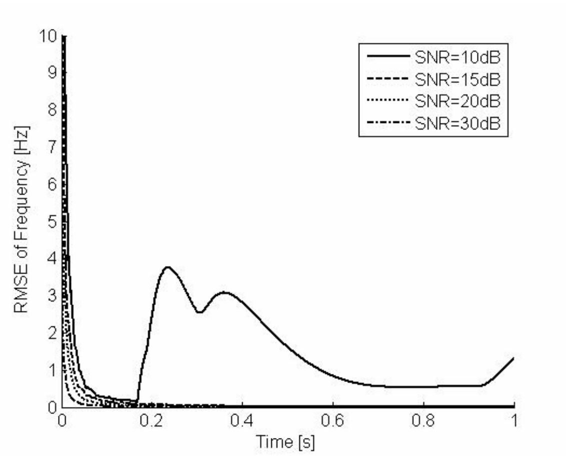

Figure 5. RMSE of estimated instantaneous frequency versus SNR.

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