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L_2 Degree reduction of interval Bézier curves using Chebyshev-Bernstein basis transformations

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Abstract - This paper presents an algorithmic approach to degree reduction of interval Bézier curves. The four fixed Kharitonov's polynomials (four fixed Bézier curves) associated with the original interval Bézier curve are obtained. The four fixed Kharitonov's polynomials (four fixed Bézier curves) associated with the approximate interval Bézier curve are also found. The algorithm is based on the matrix representations of the degree elevation and degree reduction processes. The computations are carried out by minimizing the L_2 distance between the four fixed Bézier curves P_n^i of degree n and the four fixed approximate Bézier curves Q_m^i degree m .

Keywords: computer graphics, signal and image processing, CAGD, communication systems.

I. INTRODUCTION

Computer graphics is the art and science of communicating information using images that are generated and presented through computation. This requires (a) the design and construction of models that represent information in ways that support the creation and viewing of images, (b) the design of devices and techniques through which the person may interact with the model or the view, (c) the creation of techniques for rendering the model, and (d) the design of ways the images may be preserved. The goal of computer graphics is to engage the persons visual centers alongside other cognitive centers in understanding. The Bézier curve is widely used in Image Processing, Computer Aided Geometric Design (CAGD), Computer Graphics, Pattern Recognition, Geometric Modeling, Computational Geometry, Robotics, Computer Vision and Scientific Visualization, and have many properties which are helpful for shape design. Developing more convenient techniques for designing and modifying Bézier curve is an important problem, and is also an important research issue in CAD/CAM and computer graphics technology fields. When Bézier curves are created, we often need to modify them to satisfy our design requirement. A lot of research [1-15] effort has gone into curves and surfaces in the last 30 years because of these reasons. Many sophisticated curve methods are known today-some are specialized and others are general purpose.

Degree reduction of polynomial curves and surfaces is a common process in CAGD. It consists of approximating a polynomial by another one of lower degree. This process is of great importance in geometric modelling, such as data exchange, data compression and data comparison. For example, degree reduction is needed when data are transferred from one modelling system to another and these systems have different limitations on the maximum degree of polynomials. Furthermore, it can also be used to generate a piecewise continuous lower degree approximation to a given curve or surface so as to simplify some geometric or graphical

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algorithms like intersection calculation or rendering.
Degree reduction of Bézier curves is one of the important problems in CAGD (Computer Aided Geometric Design) or the approximation theory. In general, degree reduction cannot be done exactly, so that it invokes approximation problems. Thus many papers dealing with the degree reduction have been published in the recent thirty years [16], [17].
This paper is organized as follows. Section II contains the interval Bézier curves, and section III includes the basic results where as section IV presents a numerical example, and the final section offers conclusions.

II. INTERVAL BÉZIER CURVES

An interval polynomial is a polynomial whose coefficients are interval. We shall denote such polynomials in the form $P^I(u)$ to distinguish them from ordinary (single-valued) polynomials. In general we express an interval polynomial of degree n in the form:

$$P^I(u) = \sum_{k=0}^{n} [a_k^-, a_k^+] B_k^n(u), \qquad for\ \ all\ \ u \in [0,1] \tag{1}$$

in terms of the Bernstein polynomial basis:

$$B_k^n(u) = \binom{n}{k} (1-u)^{(n-k)} u^k \quad (k = 0, 1, \ldots, n) \tag{2}$$

on $[0,1]$. Usual interval arithmetic can be applied to the interval polynomials [18].
We will define a vector-valued interval P^I in the most general terms as any compact set of points (x, y) in two dimensions. The addition of such sets is given by the Minkowski sum:

$$P_1^I + P_2^I = \{(x_1 + x_2, y_1 + y_2) | (x_1, y_1) \in P_1^I; (x_2, y_2) \in P_2^I\} \tag{3}$$

It is prudent to restrict our attention to the vector-valued intervals that are just the tensor products of scalar intervals:

$$P^I = [a^-, a^+] \times [b^-, b^+] = \{(x, y) | x \in [a^-, a^+] \ \ and \ \ y \in [b^-, b^+]\} \tag{4}$$

We occasionally use the not $([a^-, a^+], [b^-, b^+])$ instead of $([a^-, a^+] \times [b^-, b^+])$ for P^I. Such vector-valued intervals are rectangular regions in the plane, and their addition a trivial matter:

$$P_1^i + P_2^I = [a^- + c^-, a^+ + c^+] \times [b^- + d^-, b^+ + d^+] \tag{5}$$

where, $P_1^I = [a^-, a^+] \times [b^-, b^+]$ and $P_2^I = [c^-, c^+] \times [d^-, d^+]$. The extension of these ideas to vector-valued intervals in spaces of higher dimension is straightforward. An interval Bézier curve is written in the form:

$$P^I(u) = \sum_{k=0}^{n} [p_k^-, p_k^+] B_k^n(u) \tag{6}$$

where, $[p^-, p^+]$ are interval control points (rectangular intervals of the form (4)). For each $u \in [0,1]$, the value $P^I(u)$ of the interval Bézier curve (6) is a vector interval that has the following significance: For any fixed Bézier curve $P(u)$ whose control points satisfy $p_k \in [p_k^-, p_k^+]$ for $k = 0, 1, \ldots, n$ we have $P(u) \in P^I(u)$. Likewise, the entire interval Bézier curve (6) defines a region in the plane that contains all Bézier curves whose control points satisfy $p_k \in [p_k^-, p_k^+]$ for $k = 0, 1, \ldots, n$.

III. THE BASIC RESULTS

Let $\{[p_i^-, p_i^+]\}_{i=0}^n$ be a given set of interval control points which defines the interval Bézier curve:

$$P_n^I(u) = \sum_{i=0}^{n} [p_i^-, p_i^+] B_i^n(u), \qquad 0 \le u \le 1 \tag{7}$$

of degree n where,

$$B_k^j = \binom{j}{k} (1-u)^{j-k} u^k, \qquad (k = 0, 1, \ldots, j) \tag{8}$$

The problem is to find another interval point set $\{[q_i^-, q_i^+]\}_{i=0}^m$ defining the approximate interval Bézier curve:

$$Q_m^I(u) = \sum_{i=0}^{m} [q_i^-, q_i^+] B_i^m(u), \qquad 0 \le u \le 1 \tag{9}$$

of lower degree $(m < n)$ so that the weighted L_2-norm between between $P_n^I(u)$ and $Q_m^I(u)$ is a minimum.
The four fixed Kharitonov's polynomials (four fixed Bézier curves) [19] are:

$$
\begin{aligned}
P_n^1(u) &= p_0^- + p_1^- u + p_2^+ u^2 + p_3^+ u^3 + p_4^- u^4 + p_5^- u^5 + \cdots \\
&\equiv \alpha_0^1 + \alpha_1^1 u + \alpha_2^1 u^2 + \cdots + \alpha_n^1 u^n \\
P_n^2(u) &= p_0^- + p_1^+ u + p_2^+ u^2 + p_3^- u^3 + p_4^- u^4 + p_5^+ u^5 + \cdots \\
&\equiv \alpha_0^2 + \alpha_1^2 u + \alpha_2^2 u^2 + \cdots + \alpha_n^2 u^n \\
P_n^3(u) &= p_0^+ + p_1^+ u + p_2^- u^2 + p_3^- u^3 + p_4^+ u^4 + p_5^+ u^5 + \cdots \\
&\equiv \alpha_0^3 + \alpha_1^3 u + \alpha_2^3 u^2 + \cdots + \alpha_n^3 u^n \\
P_n^4(u) &= p_0^+ + p_1^- u + p_2^- u^2 + p_3^+ u^3 + p_4^+ u^4 + p_5^- u^5 + \cdots \\
&\equiv \alpha_0^4 + \alpha_1^4 u + \alpha_2^4 u^2 + \cdots + \alpha_n^4 u^n
\end{aligned} \tag{10}
$$

The four fixed Kharitonov's polynomials (four fixed approximate Bézier curves) [19] associated with the approximate interval Bézier curve are:

$$
\begin{aligned}
Q_m^1(u) &= q_0^- + q_1^- u + q_2^+ u^2 + q_3^+ u^3 + q_4^- u^4 + q_5^- u^5 + \cdots \\
&\equiv \beta_0^1 + \beta_1^1 u + \beta_2^1 u^2 + \cdots + \beta_m^1 u^m \\
Q_m^2(u) &= q_0^- + q_1^+ u + q_2^+ u^2 + q_3^- u^3 + q_4^- u^4 + q_5^+ u^5 + \cdots \\
&\equiv \beta_0^2 + \beta_1^2 u + \beta_2^2 u^2 + \cdots + \beta_m^2 u^m \\
Q_m^3(u) &= q_0^+ + q_1^+ u + q_2^- u^2 + q_3^- u^3 + q_4^+ u^4 + q_5^+ u^5 + \cdots \\
&\equiv \beta_0^3 + \beta_1^3 u + \beta_2^3 u^2 + \cdots + \beta_m^3 u^m \\
Q_m^4(u) &= q_0^+ + q_1^- u + q_2^- u^2 + q_3^+ u^3 + q_4^+ u^4 + q_5^- u^5 + \cdots \\
&\equiv \beta_0^4 + \beta_1^4 u + \beta_2^4 u^2 + \cdots + \beta_m^4 u^m
\end{aligned} \tag{11}
$$

The four fixed Bézier curves associated with the original interval Bézier curve $P_n^i(u)$ for $(i = 1,2,3,4)$ of degree n can be expressed in terms of the Bernstein and the Chebyshev polynomials:

$$
P_n^i(u) = \sum_{j=0}^{n} \alpha_j^i B_j^n(u) = \sum_{k=0}^{n} t_k^i T_k(u) \qquad (i = 1,2,3,4) \tag{12}
$$

We consider the linear transformation of the Chebyshev coefficients $t_0^i, t_1^i, \ldots, t_n^i$ into the Bernstein coefficients $\alpha_0^i, \alpha_2^i, \ldots, \alpha_n^i$ as follows:

$$
\alpha_j^i = \sum_{k=0}^{n} M_n(j,k) t_k^i \qquad (i = 1,2,3,4) \quad and \quad (j = 0,1,\ldots,n) \tag{13}
$$

The transformation above can be expressed in the following matrix form:

$$
\alpha^i = M_n t^i, \qquad (i = 1,2,3,4) \tag{14}
$$

where,

$$
\alpha^i = [\alpha_0^i \quad \alpha_1^i \quad \ldots \quad \alpha_n^i]^T
$$

$$
t^i = [t_0^i \quad t_1^i \quad \ldots \quad t_n^i]^T
$$

$$
(i = 1,2,3,4)
$$

Then the elements of the matrix $M_n(j,k)$ for $(0 \le j,k \le n)$ are given in the following formula [20]:

$$
M_n(j,k) = \frac{(2-\sqrt{(2)})\delta_k + \sqrt{(2)}}{\sqrt{\pi}} \frac{1}{\binom{n}{j}} \sum_{l=\max(0,j+k-n)}^{\min(j,k)} (-1)^{k+l} \binom{2k}{2l} \binom{n-k}{j-l} \tag{15}
$$

where,

$$
\left\{ \begin{array}{lll} \delta_k = 0 & if & k = 0 \\ \delta_k = 1 & & otherwise \end{array} \right\}
$$

The elements of the Bernstein to Chebyshev transformation matrix $M_n^{-1}(j,k)$ for $(0 \le j,k \le n)$ are given in the following formula [20]:

$$
M_n^{-1}(j,k) = \frac{\delta_j + 1}{4^{n+j}} \binom{n}{k} \sum_{l=0}^{j} (-1)^{j+l} \frac{\binom{2j}{2l}\binom{2k+2l}{k+l}\binom{2n-2k+2j-2l}{n-k+j-l}}{\binom{n+j}{k+l}} \tag{16}
$$

The weighted L_2-norm of the four fixed Bézier curves P_n^i for $(i = 1,2,3,4)$ in the Bernstein basis form is given by:

$$
\| P_n^i \|_w^2 = \int_0^1 \frac{| \sum_{j=0}^{n} \alpha_j B_j^n(u) |^2}{\sqrt{4u - 4u^2}} du \qquad (i = 1,2,3,4) \tag{17}
$$

The area under a Bernstein polynomial B_k^n for $(k = 0,1,\ldots,n)$ degree n is given by:

$$
\int_0^1 B_k^n(u) du = \frac{1}{n+1} \tag{18}
$$

The product of Bernstein polynomials of degree n and m is also Bernstein polynomial of degree $n+m$ and given by:

$$
B_j^n(u) B_k^m(u) = \frac{\binom{n}{j}\binom{m}{k}}{\binom{n+m}{j+k}} B_{j+k}^{n+m}(u) \tag{19}
$$

Simplifying equation (12) using equations (13) and (14) gives:

$$
\| P_n^i \|_w^2 = (\alpha^i)^T Q_n \alpha^i \qquad (i = 1,2,3,4) \tag{20}
$$

where,

$$
Q_n(j,k) = \frac{\Gamma(2n-j-k+\frac{1}{2})\Gamma(j+k+\frac{1}{2})}{2\Gamma(2n+1)} \binom{n}{j} \binom{n}{k}
$$

$$(j,k=0,1,\ldots,n) \qquad (21)$$

are the elements of the Gram matrix Q_n of the Bernstein basis. The matrix Q_n is real symmetric matrix, as a consequence of the symmetry of the combinatorial function. The matrix Q_n is also a positive definite matrix, as a consequence of the positivity of the left-hand side in the definition. Thus Q_n is a real symmetric positive definite matrix.

The weighted L_2-norm of the four fixed Bézier and approximate Bézier curves P_n^i and Q_m^i for $(i=1,2,3,4)$ in the Bernstein basis form is given by:

$$\| Q_m^i - P_n^i \|_w^2 = \int_0^1 \frac{| Q_m^i(u) - P_n^i(u) |^2}{\sqrt{4u-4u^2}} du$$

$$(i=1,2,3,4) \qquad (22)$$

Elevating the degree of Q_m^i from m to n using the matrix $T_{m,r}$, where, $r=n-m$ and the $(m+r+1)\times(m+1)$ matrix $T_{m,r}$ has the elements:

$$T_{m,r}(j,k) = \frac{\binom{m}{k}\binom{r}{j-k}}{\binom{m+r}{j}}$$

$$(j=0,1,\ldots,m+r) \quad \textit{and} \quad (k=0,1,\ldots,m)$$

gives:

$$Q_r^i = T_{m,r}\beta^i \qquad (23)$$

Equation (22) rewrites the curves Q_m^i of degree m as curves of degree n:

$$Q_m^i(u) = Q_r^i(u) = \sum_{j=0}^{n} \beta_j^{(r)i} B_j^n(u)$$

$$(i=1,2,3,4) \qquad (24)$$

and hence, the weighted L_2-norm is given by:

$$\| Q_n^i - P_n^i \|_w^2 = \| Q_r^i - P_n^i \|_w^2$$

$$= \int_0^1 \frac{| \sum_{j=0}^{n} (\beta_j^{(r)i} - \alpha_j^i) B_j^n(u) |^2}{\sqrt{4u-4u^2}} du$$

$$(i=1,2,3,4) \qquad (25)$$

Invoking equation (21) into the last equation gives the L_2-norm between the four fixed Bézier curves P_n^i and Q_m^i for $(i=1,2,3,4)$ in the following formula:

$$\| Q_n^i - P_n^i \|_w^2 = \| Q_r^i - P_n^i \|_w^2 = (A^i)^T Q_n A^i$$

$$(i=1,2,3,4) \qquad (26)$$

where,

$$A^i = \alpha^i - T_{m,r}\beta^i$$
$$\beta^i = [\beta_0^i \quad \beta_1^i \quad \ldots \quad \beta_m^i]^T$$
$$\alpha^i = [\alpha_0^i \quad \alpha_1^i \quad \ldots \quad \alpha_n^i]^T$$
$$(i=1,2,3,4)$$

Substituting $A^i = \alpha^i - T_{m,r}\beta^i$ in $\| Q_n^i - P_n^i \|_w^2$ and after some algebraic manipulation gives:

$$\| Q_n^i - P_n^i \|_w^2 =$$
$$(\alpha^i)^T Q_n \alpha^i - 2(\beta^i)^T T_{m,r}^T Q_n \alpha^i + (\beta^i)^T T_{m,r}^T Q_n T_{m,r} \beta^i$$
$$(i=1,2,3,4) \qquad (27)$$

The error, defined above is a function of the elements of the vectors β^i for $(i=1,2,3,4)$. To find the minimum, we use the method of least squares approximation to find the vectors $\hat{\beta}^i$ that minimizes the formula in equation (23). This process leads to the normal equations:

$$T_{m,r}^T Q_n T_{m,r} \hat{\beta}_i = T_{m,r}^T Q_n \alpha^i$$
$$(i=1,2,3,4) \qquad (28)$$

Since $T_{m,r}^T Q_n T_{m,r} = Q_m$ for $(i=1,2,3,4)$ and the matrix Q_m is positive definite. Thus $T_{m,r}^T Q_n T_{m,r}$ is invertible. Hence the normal equations are uniquely solvable and have the solutions:

$$\hat{\beta}^i = Q_m^{-1} T_{m,r}^T Q_n \alpha^i$$
$$(i=1,2,3,4) \qquad (29)$$

The four fixed Bézier curves with fixed Bézier points given in equation (25) are the best approximation curves in the least-squares sense with respect to the weighted L_2-norm.

The degree reduction with respect to the weighted L_2-norm of the four fixed Bézier curves associated with the original interval Bézier curve P_n^i for $(i=1,2,3,4)$ can be done by first transforming the Bernstein coefficients $\alpha^i = [\alpha_0^i \quad \alpha_1^i \quad \ldots \quad \alpha_n^i]^T$ for $(i=1,2,3,4)$ to the Chebyshev polynomials with coefficients $t^i = [t_0^i \quad t_1^i \quad \ldots \quad t_n^i]^T$ for $(i=1,2,3,4)$ using M_n^{-1}, and then the polynomials expressed in terms of Chebyshev polynomials are reduced to the polynomials of degree $n-1$ with coefficients $t^{(-1)i} = [t_0^{(1)i} \quad t_1^{(1)i} \quad \ldots \quad t_{n-1}^{(1)i}]^T$.

Applying the process of degree reduction r times to the four fixed Bézier curves P_n^i; for $(i=1,2,3,4)$ expressed in terms of the Chebyshev

polynomials with respect to the weighted L_2-norm, we get the polynomials of degree $n-r$ with coefficients $t^{(-r)i} = [t_0^{(r)i} \quad t_1^{(r)i} \quad \ldots \quad t_{n-r}^{(r)i}]^T$ for $(i = 1,2,3,4)$.
The r degree reduction can be written in the matrix form $t^{(-r)i} = I_{n,-r}t^i$ for $(i = 1,2,3,4)$ where, the $(n-r+1)\times(n+1)$ matrix $I_{n,-r}$ is given by:

$$I_{n,-r} = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{bmatrix} \quad (30)$$

Finally, the reduced four fixed Bézier curves expressed in terms of the Chebyshev polynomials are converted to the Bernstein coefficients using M_n . This is summarized in the following theorem.

Theorem 1: The r times degree reduction matrix $R_{m,r}$ of the four fixed Bézier curves associated with the original interval Bézier curve, and the coefficients of the four fixed reduced Bézier curves $\beta^i = \alpha^{(-r)i}$ for (i=1,2,3,4) can be calculated using the matrices M_n^{-1} , $I_{n,-r}$ and M_m as follows:

$$R_{m,r} = M_m I_{n,-r} M_n^{-1}$$

$$\beta^i = \alpha^{(-r)i} = R_{m,r}\alpha^i$$

$$(i = 1,2,3,4)$$

However, the r times degree reduction interval error $(E_w^I)^2$ of the original interval Bézier curve with respect to the weighted L_2-norm is given in the following theorem.

Theorem 2: The r times degree reduction interval error $(E_w^I)^2$ of the original interval Bézier curve with respect to the weighted L_2-norm is given by:

$$([E_w^-, E_w^+])^2 = [\min(E_w^i)^2, \max(E_w^i)^2]$$

$$(i = 1,2,3,4)$$

where,

$$(E_w^i)^2 = (\alpha^i)^T F_{m,r}^i \alpha^i$$

$$(i = 1,2,3,4)$$

and

$$F_{m,r}^i = Q_n[I - T_{m,r}(T_{m,r}^T Q_n T_{m,r})^{-1} T_{m,r}^T Q_n]$$

$(E_w^i)^2$ for $(i = 1,2,3,4)$ are the r times degree reduction errors of the four fixed Bézier curves associated with the original interval Bézier curve with respect to the weighted L_2-norm.
Finally, the new interval vertices of the new interval polygon can be obtained as follows:

$$[q_k^-, q_k^+] = [\min(\beta_k^i), \max(\beta_k^i)]$$

$$(k = 0,1,\ldots,n-r) \quad \textit{and} \quad (i = 1,2,3,4) \quad (31)$$

IV. NUMERICAL EXAMPLE

Consider the interval Bézier curve defined by the four interval control points:

$$[p_0^-, p_0^+] = ([0.6000, 0.7500], [1.0000, 1.1000])$$

$$[p_1^-, p_1^+] = ([1.7500, 2.0000], [2.0000, 2.2500])$$

$$[p_2^-, p_2^+] = ([3.1000, 3.4000], [2.4500, 2.6000])$$

$$[p_3^-, p_3^+] = ([2.3500, 2.5000], [0.8500, 1.0000])$$

It is required to reduce the degree of the interval Bézier curve defined by them.
The new interval vertices $\{[q_i^-, q_i^+]\}_{i=0}^2$ of the new interval polygon are obtained as explained in section III:

$$[q_0^-, q_0^+] = ([0.5047, 0.6969], [0.9437, 1.0734])$$

$$[q_1^-, q_1^+] = ([2.8250, 3.3125], [2.8125, 3.1750])$$

$$[q_2^-, q_2^+] = ([2.4031, 2.5953], [0.8766, 1.0562])$$

The interval error $(E_w^I)^2$ with respect to the weighted L_2-norm is:

$$([E_w^-, E_w^+])^2 = ([0.0006, 0.0018], [0.0001, 0.0006])$$

$Q_2^I(u)$ is the approximate interval Bézier curve of the given interval Bézier curve $P_3^I(u)$, so that the weighted L_2-norm between between $P_3^I(u)$ and $Q_2^I(u)$ is a minimum.

Simulation results in Figure(1) shows the envelopes of the original interval Bézier curve and the reduced interval Bézier curve, respectively.

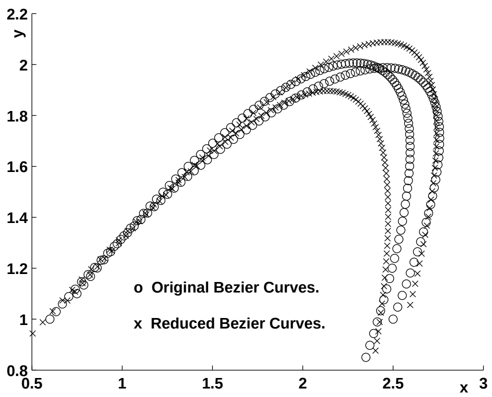

Fig.1:The original and reduced Bezier curve envelopes.

V. CONCLUSIONS

An algorithmic approach to degree reduction of interval Bézier curves is presented in this paper. The four fixed Kharitonov's polynomials (four fixed Bézier curves) associated with the original interval Bézier curve are obtained. The four fixed Kharitonov's polynomials (four fixed Bézier curves) associated with the approximate interval Bézier curve are also found. The algorithm is based on the matrix representations of the degree elevation and degree reduction processes. The computations are carried out by minimizing the L_2 distance between the four fixed Bézier curves P_n^i of degree n and the four fixed approximate Bézier curves Q_m^i of degree m.

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