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Estimation of Noisy Sinusoids Instantaneous Frequency by Kalman Filtering

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Abstract – The paper adresses the problem of estimating the instantaneous frequency of discrete time sinusoids imbedded in Gaussian noise. The proposed method is based on a model of the signal phase as a polynomial. This approach offers the opportunity to represent these signals by an adequate state space model and to apply standard Kalman filtering procedures in view to estimate the parameters of the phase polynomial. Procedure simulations were made on linear chirp sinusoids and are consistent with the theoretical approach. The paper presents the most important results.

Keywords: instantaneous frequency, polynomial phase, chirp signal, Kalman filter

I. INTRODUCTION

In order to estimate the instantaneous frequency of sinusoids corrupted by noise, we use an adequate model of the signal with emphasis on its instantaneous phase. This section introduces the model of a polynomial phase sinusoid and the use of state space description in Kalman estimation of its parameters.

Take a non-stationary continuous-time signal, $s(t)$, given by:

$$s(t) = A\cos\Phi(t) \quad (1)$$

where A is constant. For non-stationary signals, (signals whose spectral contents vary with time) the frequency at a particular instant of time is described by the concept of instantaneous frequency. The instantaneous frequency $f_i(t)$ of this signal is [1] ÷ [6]:

$$f_i(t) = \frac{1}{2\pi}\frac{d\Phi(t)}{dt} \quad (2)$$

We limit the discussion at the signals, whose phase $\Phi(t)$ is an M-order polynomial,

$$\Phi(t) = \sum_{k=0}^{M} a_k t^k \quad (3)$$

The instantaneous frequency for these signals becomes:

$$f_i(t) = \frac{1}{2\pi}\sum_{k=1}^{M} k a_k t^{k-1} \quad (4)$$

The discrete-time signals, $s[n]$, where n is the normalized time, are given by:

$$s[n] = A\cos\Phi[n] \quad (5)$$

where $\Phi[n]$ is a polynomial defined as:

$$\Phi[n] = \sum_{k=0}^{M} a_k n^k \quad (6)$$

At the normalized moment of time n, the instantaneous frequency is given by:

$$f_i[n] = \frac{1}{2\pi}\sum_{k=1}^{M} k a_k n^{k-1} \quad (7)$$

It can be seen that if the polynomial's coefficients $a_1 \div a_M$, which describe the phase, are known it is possible to achieve the measurement of the instantaneous frequency for this class of signals.

There are two classes of methods to establish the polynomial's coefficients or the instantaneous frequency:

i) nonparametric methods which resort to time-frequency representation [1], [5] and

ii) parametric methods [2], [3], [6], [9], based on credible model for the signal in which the parameter's values are determining it.

In this paper we will find a states space model for the signal so as to be able to resort to Kalman filtering for the parameters determination.

As a rule of thumb, if the state vector $\mathbf{X}[n]$ describes the state of the system, which is determined by measured values summed in $\mathbf{Y}[n]$, we have [7]:

$$\mathbf{X}[n+1] = \mathbf{A}\mathbf{X}[n] + \mathbf{N}[n] \quad (8)$$

$$\mathbf{Y}[n] = \mathbf{B}\mathbf{X}[n] + \mathbf{W}[n] \quad (9)$$

In these relations $\mathbf{N}[n]$ is playing the role of the excitation, but it can represent also only a noise. $\mathbf{W}[n]$ is the vector of noise in the measured signal. $\mathbf{A}$ is the transition matrix and $\mathbf{B}$ is the measurement matrix.

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Based on eq. (8) and (9), we will find a model for polynomial phase signals imbedded in additive white Gaussian noise with zero-mean value.

II. THE MODEL OF THE NOISY SINUSOID

If $s[n]$ is a signal given by (5), the associated analytical signal is $A\exp\{j\Phi[n]\}$. In our measurement scheme, the signal values are corrupted by additive white Gaussian noise $w[n]$, zero-mean and variance σ^2. Let's consider the associated analytical noise $w[n]$ given by:

$$w[n] = w_R[n] + jw_I[n] \tag{10}$$

with $w_R[n]$ and $w_I[n]$, the real part and the imaginary part of the analytical noise. If both parts are not correlated between them, having the same variance, we can write:

$$E\{w_R[n]w_R[n+k]\} = \frac{\sigma^2}{2}\delta[k] \tag{11}$$

$$E\{w_I[n]w_I[n+k]\} = \frac{\sigma^2}{2}\delta[k] \tag{12}$$

$$E\{w_R[n]w_I[n+k]\} = 0, \quad \forall k \in Z \tag{13}$$

where $E\{\cdot\}$ is the expectation operator. An analytical signal having these properties is called "cyclic" noise [8]. The measured signal $y[n]$ is obtained by the addition:

$$y[n] = A\exp\{j\Phi[n]\} + w[n] \tag{14}$$

We consider the last vectorial addition in terms of polar components of complex signal in (14):

$$\mathbf{Y}[n] = \begin{bmatrix} |y[n]| \\ Arg\{y[n]\} \end{bmatrix} \tag{15}$$

As is shown in [7], if the signal-to-noise ratio (SNR) in the measured signal $y[n]$ exceeds 13dB, the noise real part affects only the amplitude A, whereas the phase $\Phi[n]$ is affected by the imaginary part of the "cyclic" noise. Eq. (14) can be written now in terms of amplitude and phase as:

$$\begin{bmatrix} |y[n]| \\ Arg\{y[n]\} \end{bmatrix} = \begin{bmatrix} A \\ \Phi[n] \end{bmatrix} + \begin{bmatrix} v_R[n] \\ v_I[n]/A \end{bmatrix} \tag{16}$$

III. STATE VECTOR AND TRANSITION EQUATIONS

The values of an M-order polynomial $P(x)$, can be expressed by the Taylor series expansion [7]:

$$P(x_0+\Delta x) = \sum_{k=0}^{M}\frac{(\Delta x)^k}{k!}P^{(k)}(x_0); \forall x_0, \forall \Delta x \in R \tag{17}$$

viewing that all derivatives having the order higher than M are zero. For the l-order derivative of the polynomial $P^{(l)}(x)$ can be used the following series expansion:

$$P^{(l)}(x_0+\Delta x) = \sum_{k=l}^{M}\frac{(\Delta x)^k}{(k-l)!}P^{(k)}(x_0); \quad \forall x_0, \forall \Delta x \in R, l = \overline{1,M} \tag{18}$$

Replacing $P(x)$ by $\Phi[n]$, x_0 with n and Δx by 1, we have:

$$\Phi[n+1] = \sum_{k=0}^{M}\frac{1}{k!}\Phi^{(k)}[n] \tag{19}$$

$$\Phi^{(l)}[n+1] = \sum_{k=l}^{M}\frac{1}{(k-l)!}\Phi^{(k)}[n] \quad l = \overline{1,M} \tag{20}$$

The state vector $\mathbf{X}[n]$ is given by the amplitude of the sinusoid, the phase and the first M derivatives of the phase:

$$\mathbf{X}[n] = \begin{bmatrix} A & \Phi[n] & \Phi^{(')}[n] & \dots & \Phi^{(M)}[n] \end{bmatrix}^T \tag{21}$$

having an $(M+2)\times 1$ size. The state at the moment $n+1$ can be written as:

$$\mathbf{X}[n+1] = \\ = \begin{bmatrix} A & \Phi[n+1] & \Phi^{(')}[n+1] & \Phi^{('')}[n+1] & \dots & \Phi^{(M)}[n+1] \end{bmatrix}^T \tag{22}$$

For two consecutive moments, the relation between the two states is derived from (19) and (20):

$$\begin{bmatrix} A \\ \Phi[n+1] \\ \Phi^{(')}[n+1] \\ \Phi^{('')}[n+1] \\ \vdots \\ \Phi^{(M)}[n+1] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots 0 \\ 0 & 1 & \frac{1}{1!} & \frac{1}{2!} & \dots \frac{1}{M!} \\ 0 & 0 & 1 & \frac{1}{1!} & \dots \frac{1}{(M-1)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots 1 \end{bmatrix} \cdot \begin{bmatrix} A \\ \Phi[n] \\ \Phi^{(')}[n] \\ \Phi^{('')}[n] \\ \vdots \\ \Phi^{(M)}[n] \end{bmatrix} \tag{23}$$

We obtain a transition equation. Comparing (23) with (8), we find that the $(M+2)\times(M+2)$-size transition matrix is:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \frac{1}{1!} & \frac{1}{2!} & \dots & \frac{1}{M!} \\ 0 & 0 & 1 & \frac{1}{1!} & \dots & \frac{1}{(M-1)!} \\ 0 & 0 & 0 & 1 & \dots & \frac{1}{(M-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{24}$$

and $\mathbf{N}[n]$ is the null vector:

$$\mathbf{N}[n]=\mathbf{0} \tag{25}$$

IV. THE MEASUREMENT EQUATION

The measurement vector has two components having, therefore a (2×1) size. Because the state vector has a $(M+2)\times1$ size, the measurement matrix $\mathbf{B}$ is $2\times(M+2)$ size. As shows [10], the measured output variables $\mathbf{Y}[n]$ can be obtained from (16) by:

$$\mathbf{Y}[n]=\begin{bmatrix} |y[n]| \\ Arg\{y[n]\} \end{bmatrix}=$$

$$=\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \end{bmatrix}\begin{bmatrix} A \\ \Phi[n] \\ \Phi^{(')}[n] \\ \Phi^{('')}[n] \\ \vdots \\ \Phi^{(M)}[n] \end{bmatrix}+\begin{bmatrix} v_R[n] \\ \dfrac{v_I[n]}{A} \end{bmatrix} \tag{26}$$

This is the measurement equation. Comparing (26) and (9) we find that the measurement matrix $2\times(M+2)$ size is:

$$\mathbf{B}=\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{27}$$

The vector $\mathbf{W}[n]$, must be considered as:

$$\mathbf{W}[n]=\begin{bmatrix} v_R[n] \\ \dfrac{v_I[n]}{A} \end{bmatrix} \tag{28}$$

V. CONNECTIONS BETWEEN STATE MATRIX AND PHASE POLYNOMIAL COEFFICIENTS

We determined a model of the signal (5) in the states space by (23) and (26). We can determine the state vector $\mathbf{X}[n]$ using the Kalman filtering. We want to obtain the way of getting the coefficients $a_1 \div a_M$ or even $a_0 \div a_M$.

To know the states means to know the following: A, $\Phi[n]$ and $\Phi^{(l)}[n]$, $l=\overline{1,M}$. If we know $\Phi[n]$, $\Phi^{(l)}[n]$ and replacing the value for stationary regime, for a given n in (19) and (20), we get an equation like (6):

$$\sum_{k=l}^{M}\sum_{m=k}^{M}\frac{(m-k)!}{(k-l)!}a_m n^{m-k}=\Phi^{(l)}[n],\ n\text{ given},\ l=\overline{0,M} \tag{29}$$

Solving the M linear equations system (29) we determine the $a_1 \div a_M$ coefficients. For a given n, the solution of (29) is succesively calculated, starting with the last coefficient a_M:

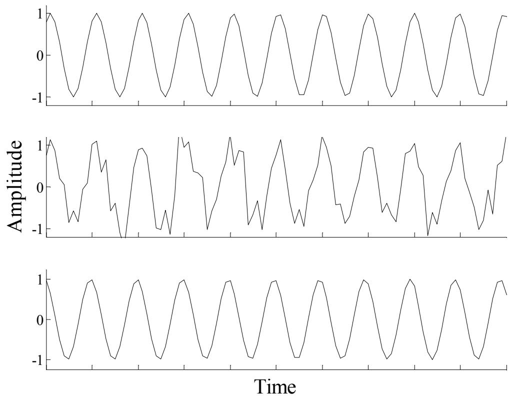

Fig. 1 The use of the polynomial phase model in view to apply a Kalman filter procedure to a noisy sinusoid.

$$a_M=\frac{1}{M!}\Phi[n]$$

$$a_{M-l}=\frac{1}{(M-l)!}\left[\Phi^{(M-k)}[n]-\sum_{k=0}^{l-1}\frac{(M-k)!}{k!}n^k a_{M-k}\right], \tag{30}$$

$$l=\overline{1,M}$$

The instantaneous frequency, is computed directly from $\Phi^{(')}[n]$, as:

$$f_i[n]=\frac{1}{2\pi}\Phi^{(')}[n] \tag{31}$$

VI. EXPERIMENTAL RESULTS

In order to implement the states space model of a noisy constant amplitude sinusoid introduced before we used linear chirp sinusoids corrupted by noise. Hilbert transformation followed by modulus and phase calculation is applied to noisy sinusoid to obtain the Cartesian coordinates decomposition of eq. (15). These data represent the measured input vector for a Kalman filtering algorithm based on one-step prediction, which is implemented in MATLAB.

The reference signal, as was mentioned before is a constant amplitude linear frequency modulated sinusoid. The phase of this signal is described by a second-order polynomial, $M=2$ in eq. (3). The sampling frequency is 5000Hz. During one second, the instantaneous frequency of chirp sinusoid changes linearly between 100Hz and 900Hz. A part from this signal is represented in the first graph of Fig. 1. The second graph in Fig 1 shows the measured signal obtained by the addition of a zero-mean and variable variance white Gaussian noise to the reference signal. The signal-to-noise ratio (SNR) for this example is 10dB. Finally, the last image in Fig. 1 shows the result of Kalman filtering algorithm application to second image signal.

To give a better understanding of Kalman filter action on polynomial phase signals, Fig. 2 shows the results of instantaneous frequency $f_i(t)$ measurements performed on signals represented in Fig.1. In this case $\mathrm{SNR}=23\,\mathrm{dB}$. If, for the reference and measurement

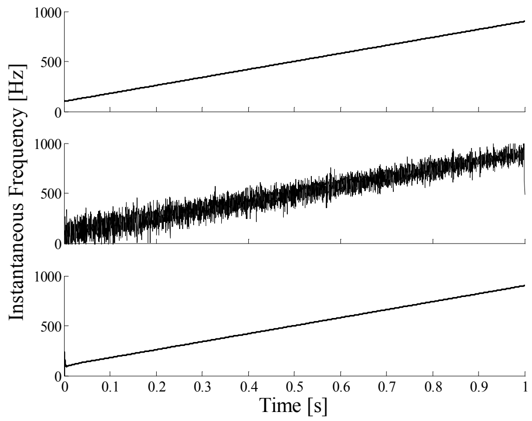

Fig. 2 The measurement of the instantaneous frequency of a polynomial phase noisy sinusoid by Kalman filtering.

signals, $f_i(t)$ established by (2) are represented in the first two images, the Kalman filter output shown in the last image, gives the estimation of $f_i(t)$ computed in (31), where $\Phi^{(\cdot)}[n]$ is given by the third element of the state vector $\mathbf{X}[n]$. Fig. 2 is persuasive on the efficiency of Kalman filtering of polynomial phase noisy sinusoids in order to establish the instantaneous frequency.

In order to give an objective measure of the quality of the instantaneous frequency estimation by our method we define the *RMS Frequency Error*, $\Delta f_i^{RMS}[n]$ as

$$\Delta f_i^{RMS}[n] = \sqrt{E\left\{\left(f_i^{ref}[n] - f_i^{est}[n]\right)^2\right\}} \qquad (32)$$

where $f_i^{ref}[n]$ is the instantaneous frequency of the reference, and $f_i^{est}[n]$, the intantaneous value estimated by Kalman filtering. $E\{\cdot\}$ denotes the statistical expectation operator.

Fig. 3 and Fig. 4 use $\Delta f_i^{RMS}[n]$ to evaluate the performances of Kalman filter in instantaneous frequency estimation. The first experiment in Fig. 3 is made with SNR=13dB and shows that the maximum performances are obtained when the state vector

Fig. 3 The dependency of RMS Frequency Error on the order M+2 of the state vector **X**[n] used in Kalman filter.

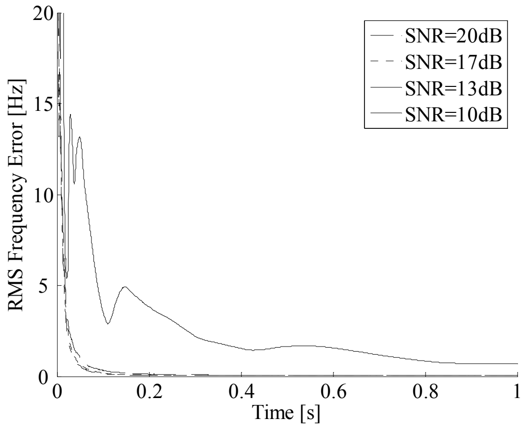

Fig. 4 The dependency of RMS Frequency Error on the SNR of the input noisy sinusoid.

$\mathbf{X}[n]$ is of minimum order: M+2=4. As about the Fig. 4, it was drawn for M=2 case and confirms that the method performs well as long as SNR exceeds 13dB.

VII. CONCLUSIONS

The paper gives the state space model of polynomial phase signals with good opportunities in instantaneous frequency estimation. The Kalman filter implemented on this model presents very encouraging results in the case of noisy linear frequency variation sinusoids.

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