Buletinul Ştiinţific al Universităţii "Politehnica" din Timişoara

Seria ELECTRONICĂ şi TELECOMUNICAŢII
TRANSACTIONS on ELECTRONICS and COMMUNICATIONS

Tom 49(63), Fascicola 2, 2004

Phase approximation using signals affected by random perturbations

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Abstract

The goal of this paper is to give a comparison between two methods for phase approximations: non-compact gain technique for linear frequency domain and the approach based on logarithmic sampling of gain for logarithmic frequency domain, using signals affected by random perturbations. A comparison of the behavior of these algorithms, considering signals affected by perturbations, respectively signals that are not affected by perturbations, will be also presented. For this purpose we first recall Hilbert transform and Bode relationships, then the two methods will be discussed. Numerical examples are provided to emphasize the advantages and disadvantages of each method and computer simulations performed using Matlab are also presented.

Keywords: phase approximation, logarithmic sampling, linear sampling, Hilbert transform, Bayard-Bode relationships.

1. INTRODUCTION

The non-compact gain technique cannot be employed when the gain characteristic has slopes different from zero at zero and at high frequency. Since the Bode transfer functions do not satisfy the last requirement, in order to overpass this inconvenient, we have proposed the modified Bode transfer functions to be used in implementation.

We will present three cases of how signals are affected by random perturbations:

1. signal affected by a complex random perturbation;
2. system function affected by a real random perturbation;
3. parameters of the Bode transfer function, respectively of the modified Bode transfer function affected by a real random perturbation.

The proposed methods are then tested on some numerical examples. Our analysis will be concentrated on minimum-phase functions, since the experimental results can be very easily applied to non minimum-phase functions [6].

Hilbert transform and Bayard-Bode relationships [1] have been recognized as very important methods in circuit theory, communications and control science. Their sampled derivations have been encountered in different applications from science and engineering. In some situations the domain is restricted or other explicit conditions are imposed. A critical issue is related to the singularities involved in the Hilbert transform computation, since we are confronted with an improper integral (Section 2). If the integral cannot be evaluated in a closed form, as it is the case with discrete input data, numerical implementation is in general complicated [2], as localized errors should lead to localized errors. Hilbert transform has the advantage of not requiring derivatives, but the serious disadvantage that it is not a bounded operator from L_r to L_∞. To solve the problem, different approaches for gain-phase relationships in logarithmic frequency domain have been proposed. A suitable change of variable can give the bounded operator (5) from L_r to L_∞ for any $r \geq 1$ [3].

The goal of this paper is to give a comparison between linear and logaritmic frequency domain phase approximations, using as test signals, signals that are affected by random perturbations. There is also presented a comparison of these algorithms, considering signals that are affected by perturbations, respectively signals that are not affected by perturbations.

The paper is organised as follows. In Section 2 we shortly remind Hilbert transform and Bode relationships. First we will discuss the logarithmic sampling of gain approach in Section 3, then in Section 4 the non-compact gain technique [4] is addressed. Furthermore, we derive the modified Bode transfer functions (Section 5) to be used in implementation. Finally it results a comparison based on numerical examples (Section 6) and we will drag conclusions (Section 7).

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2. BAYARD-BODE RELATIONSHIPS AND HILBERT TRANSFORM

The Bayard-Bode relations method is based on the fact that the transform

$$H(j\omega) = R(\omega) + jI(\omega), \quad (1)$$

of a causal function $h(t)$ is uniquely determined in terms of $R(\omega)$ or $I(\omega)$ (subject to an arbitrary reactance value if determined from $R(\omega)$ and to an arbitrary real value, if determined from $I(\omega)$) [1]. Proofs based on Cauchy's residue theorem [13] or on convolution [6] establish

$$\begin{aligned} R(\omega) &= R(\infty) - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{I(y)}{y-\omega}dy = \\ &= R(\infty) - \frac{2}{\pi}\int_{0}^{\infty}\frac{yI(y)-\omega I(\omega)}{y^2-\omega^2}dy \end{aligned} \quad (2)$$

$$I(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R(y)}{y-\omega}dy = \frac{2\pi}{\omega}\int_{0}^{\infty}\frac{R(y)-R(\omega)}{y^2-\omega^2}dy \quad (3)$$

One can easily obtain the gain-phase relationships (or the Bayard-Bode relations) from (2) and (3) directly by taking logarithms [6], after fulfilling the requirements needed to satisfy the right half plane analyticity conditions of the Hilbert transform, i.e. the stable and minimum phase conditions. Under the assumption that $H(s)$ is not only analytic, but has no zeros for $\mathrm{Re}(s) \geq 0$, then:

$$\ln(H(j\omega)) = \alpha(\omega) + j\beta(\omega) \quad (4)$$

will be also analytical in the right-hand plane. Thus the phase $\beta(\omega)$ (in nepers), using a change of variable $u = \ln(y/\omega_c)$, where ω_c is a normalizing frequency, will be:

$$\begin{aligned} \beta(\omega) &= \frac{2\omega}{\pi}\int_{0}^{\infty}\frac{\alpha(y)-\alpha(\omega)}{y^2-\omega^2}dy = \\ &\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\alpha(\omega_c e^u)-\alpha(\omega_c)}{e^u-e^{-u}}du = \\ &= \frac{1}{\pi}\int_{0}^{\infty}\left(\frac{d}{du}\alpha(\omega_c e^u)\right)\ln\left(\coth\frac{|u|}{2}\right)du \end{aligned} \quad (5)$$

3. PHASE APPROXIMATION IN LOGARITHMIC FREQUENCY DOMAIN

The scope is to find a phase approximation from the gain samples, given at equally spaced points on the logarithmic frequency domain:

$$\beta(\omega) \approx \sum_{p\in N}\Gamma_p[\alpha(\omega\Delta^p) - \alpha(\omega\Delta^{-p})], \quad \Delta > 1 \quad (6)$$

Using quadrature formulae, several approximations results. Here we shall consider for study that one derived from Simpson approach (the parabolic rule):

$$\begin{aligned} \beta_S(\omega) = \frac{1}{\pi}\Bigg[&\alpha(\omega\Delta) - \alpha\left(\frac{\omega}{\Delta}\right)\Bigg] + \\ &+\frac{2\ln\Delta}{3\pi}\Bigg[\frac{\alpha(\omega\Delta)-\alpha(\omega\Delta^{-1})}{\Delta-\Delta^{-1}} + \\ &+4\frac{\alpha(\omega\Delta^2)-\alpha(\omega\Delta^{-2})}{\Delta^2-\Delta^{-2}} + 2\frac{\alpha(\omega\Delta^3)-\alpha(\omega\Delta^{-3})}{\Delta^3-\Delta^{-3}} + \\ &+\ldots+4\frac{\alpha(\omega\Delta^{k-1})-\alpha(\omega\Delta^{1-k})}{\Delta^{k-1}-\Delta^{1-k}} + \\ &+2\frac{\alpha(\omega\Delta^k)-\alpha(\omega\Delta^{-k})}{\Delta^k-\Delta^{-k}}\Bigg] \end{aligned} \quad (7)$$

or

$$\beta_S(\omega) = \sum_{p\in Z} S_p\alpha(\omega\Delta^p),$$

$$S_p = S_{-p} = \begin{cases} \dfrac{1}{\pi}\left(1 + \dfrac{2/3\ln\Delta}{\Delta - 1/\Delta}\right), & p = 1 \\ \dfrac{8\ln\Delta}{3\pi\left(\Delta^p - \Delta^{-p}\right)}, & p = \pm2,\ldots\pm2m \\ \dfrac{4\ln\Delta}{3\pi\left(\Delta^p - \Delta^{-p}\right)}, & p = \pm3,\ldots\pm(2m-1) \\ \dfrac{2\ln\Delta}{3\pi\left(\Delta^p - \Delta^{-p}\right)}, & p = \pm(2m+1) \\ 0, & otherwise \end{cases} \quad (8)$$

4. PHASE APPROXIMATION IN LINEAR FREQUENCY DOMAIN

The formula between the imaginary and real parts of a complex function of real frequency as expressed in equation (3) can be rewritten in many ways [4]. By integrating the right member of (3) by parts one can find:

$$I(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}R'(y)\ln\left|\frac{y+\omega}{y-\omega}\right|dy \quad (9)$$

provided

$$\lim_{y\to\infty}\frac{R(y)}{y}=0 \qquad (10)$$

Alternatively, we can continue by integrating the right member of (9) by parts, i.e. a double integration by parts of the right member of (3) and the integrand will be:

$$R''(y)\left\{2\omega-\omega\ln\left|\omega^2-y^2\right|-y\ln\left|\frac{y+\omega}{y-\omega}\right|\right\} \qquad (11)$$

provided

$$\lim_{y\to\infty}\frac{R(y)}{y}=0 \qquad (12)$$

and

$$\lim_{y\to\infty}R'(y)y<\infty \qquad (13)$$

Previous relationships are seldom integrated analytically and in practice it is customary to use approximations to find the relationship between phase and gain. An idea is to use straight-line segments so that the second derivative $\alpha''(y)$ is a set of impulses [2]. Gain functions will satisfy the following:

- Second derivative consists of groups of 2 impulses;
- Each group has a positive impulse at the origin and a negative impulse at a frequency ω_n;
- Only positive ω_n'n need to be considered.

Thus the second derivative of the gain is given as

$$\alpha''(y)\approx\sum_n a_n\left[\delta(y)-\delta(y-\omega_n)\right] \qquad (14)$$

It follows succesivelly that[2]:

$$\beta(\omega)\overset{(3)}{=}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\alpha(y)}{y-\omega}dy\overset{(9)}{=}\frac{1}{\pi}\int_{-\infty}^{\infty}\alpha'(y)\ln\left|\frac{y+\omega}{y-\omega}\right|dy\overset{(10)}{=}$$
$$\frac{1}{\pi}\int_{-\infty}^{\infty}\alpha''(y)\times\left\{2\omega-\omega\ln\left|\omega^2-y^2\right|-y\ln\left|\frac{y+\omega}{y-\omega}\right|\right\}= \qquad (15)$$
$$=\cdots\equiv\beta_a(\omega)$$

Finally we have:

$$\beta(\omega)\approx\beta_a(\omega)=\frac{1}{\pi}\sum_n a_n\omega_n\Phi\left(\frac{\omega}{\omega_n}\right) \qquad (16)$$

where

$$\Phi(v)=(v+1)\ln|v+1|+(v-1)\ln|v-1|-2v\ln|v|$$

Remarks:

1. The a_n numbers are determined by a broken-line approximation to the gain-versus-arithmetic-frequency characteristic.
2. This procedure cannot be employed when the gain characteristic has slopes different from zero at zero and at high frequency.
3. The non-compact support gain method can be easily extended to broken-parabolic (or higher order curve approximation.

5. MODIFIED BODE TRANSFER FUNCTIONS

Previous attempts to test the phase approximations approaches have used the Bode transfer functions [1]

$$H(s)=\frac{1|}{|s}+\frac{1|}{|sK^2}+\frac{1|}{|s/H}+\frac{1|}{|1}=$$
$$=\frac{1}{s+\cfrac{1}{sK^2+\cfrac{1}{s/H+1}}}= \qquad (17)$$

The magnitude of the frequency response $|H(j\omega)|$ is given by:

$$\frac{\sqrt{(-K^2\omega^2)^2+(K^2H\omega)^2}}{\sqrt{(-K^2H\omega^2+H)^2+[K^2\omega^3-(H+1)\omega]^2}}$$

and the gain $\alpha(\omega)$ has slopes different from zero at high frequency.

We shall slightly modify the Bode transfer functions as follows:

$$H(s)=\frac{As+B}{s+\cfrac{1}{sK^2+\cfrac{1}{s/H+1}}} \qquad (18)$$

and now we are looking what requirements should satisfy the parameters A, B, K and H such that the gain has zero slopes at zero and at high frequency. We have the following expresions for $H(s)$, $H(j\omega)$ and $|H(j\omega)|$ respectively:

$$\frac{AK^2s^3+(AH+B)K^2s^2+(A+BK^2)Hs+BH}{K^2s^3+K^2Hs^2+(H+1)s+H}$$

$$\frac{[BH-(AH+B)K^2\omega^2]+j[(A+BK^2)H\omega-AK^2\omega^3]}{(H-K^2H\omega^2)+j[(H+1)\omega-K^2\omega^3]}$$

[2] An extended form of $\beta(\omega)$ can be found in [11]

$$\frac{\sqrt{[BH-(AH+B)K^2\omega^2]^2+[(A+BK^2)H\omega-AK^2\omega^3]^2}}{\sqrt{(H-K^2H\omega^2)^2+[(H+1)\omega-K^2\omega^3]^2}}$$

Thus

$$\alpha(\omega)=\frac{1}{2}\ln U(\omega)-\frac{1}{2}\ln V(\omega),$$

where

$$U(\omega)=[BH-(AH+B)K^2\omega^2]^2+$$
$$+[(A+BK^2)H\omega-AK^2\omega^3]^2;$$

$$V(\omega)=(H-K^2H\omega^2)^2+[(H+1)\omega-K^2\omega^3]^2$$

The gain slope is given by[3]:

$$\alpha'(\omega)=\frac{U'(\omega)V(\omega)-V'(\omega)U(\omega)}{2U(\omega)V(\omega)}=$$
$$=\frac{-(2A^2+B^2K^2)\omega^9+\cdots+(\cdots)H^2\omega}{2A^2K^8\omega^{12}+\cdots+B^2H^4} \qquad (19)$$

Now,

1. From $\lim_{\omega\to\infty}\alpha'(\omega)=0$, we need $A^2K^4\cdot K^4\neq 0$;
2. From $\lim_{\omega\to 0}\alpha'(\omega)=0$, it follows $H^2\cdot B^2H^2\neq 0$.

Consequently, the modified Bode transfer functions should satisfy the requirements:

$$A\cdot B\cdot K\cdot H\neq 0 \qquad (20)$$

To see the behaviour of the phase approximation algorithms under random perturbation conditions in the logarithmic frequency domain, we have three cases:

1. signal affected by a complex random perturbation

$$H_{test}(j\omega)=H(j\omega)+noise\_complex \qquad (21)$$

2. system function affected by a real random perturbation

$$|H_{test}(j\omega)|=|H(j\omega)|+noise\_real \qquad (22)$$

3. parameters of the Bode transfer function, respectively of the modified Bode transfer function affected by a real random perturbation

$$H_{test}=H+noise\_real \qquad (23)$$

$$K_{test}=K+noise\_real \qquad (24)$$

6. SIMULATIONS

Now we are going to compare the given approaches.

A. *Logarithmic Frequency Domain*

For logarithmic frequency domain, the selected transfer function is:

$$H(s)=\cfrac{1}{s+\cfrac{1}{4s+\cfrac{1}{\frac{s}{2}+1}}} \qquad (25)$$

where we used the Bode transfer function (17), considering $K=H=2$. The phase of the selected transfer function (i) is almost constant for $\omega<0.01$ and $\omega>10$ [1], consequently the interval of interest in our experiments was $\omega\in[0.01,10]$. We select $\Delta=\sqrt{2}$ as sample ratio. Three plots are shown for different number of samples: $k=5$ (ii), $k=9$ (iii) and $k=17$ (iv).

If noise is present, then it can affect the quality of phase approximation. The phase (v) and phase approximations for different number of samples: $k=5$ (vi), $k=9$ (vii) and $k=17$ (viii) using as test signal one that is affected by random perturbations are also ploted.

1. signal affected by a complex random perturbation

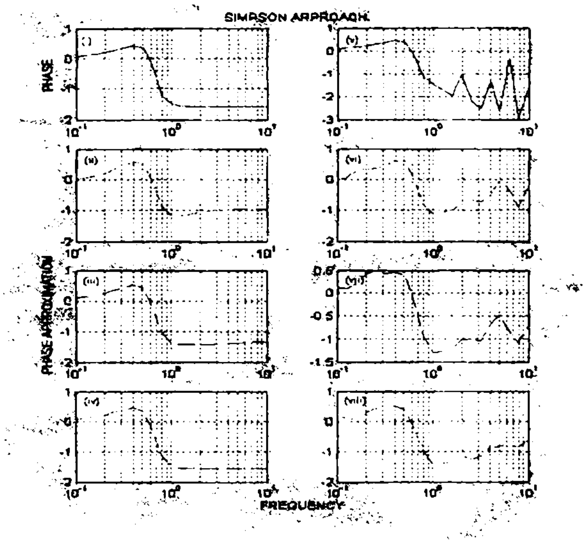

Fig. 1. Phase (i) and phase approximation (ii), (iii), (iv) for the transfer function (25); phase (v) and phase approximation (vi), (vii), (viii) using signal affected by a complex random perturbation

2. system function affected by a real random perturbation

[3] An extended form of $\alpha(\omega)$ can be found in [8]

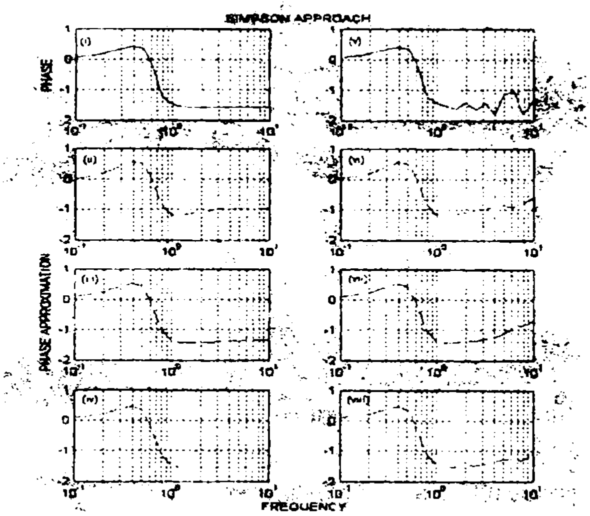

Fig. 2. Phase (i) and phase approximation (ii), (iii), (iv) for the transfer function (25); phase (v) and phase approximation (vi), (vii), (viii) using system function affected by a real random perturbation

3. parameters of the Bode transfer function affected by a real random perturbation

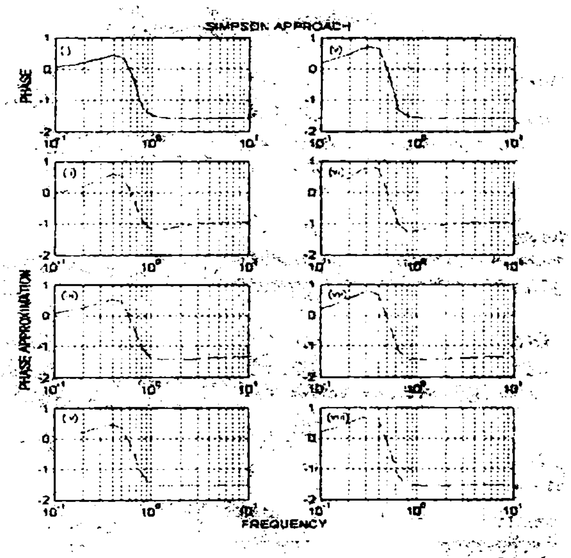

Fig. 3. Phase (i) and phase approximation (ii), (iii), (iv) for the transfer function (25); phase (v) and phase approximation (vi), (vii), (viii) using parameters affected by a real random perturbation

B. Linear Frequency Domain

For linear frequency domain, the selected transfer function is:

$$H(s)=\frac{s+1}{\dfrac{1}{s+\dfrac{1}{s+1}}} \qquad (26)$$

where we used the modified Bode transfer function (18), considering $A=B=K=H=1$. Relation (26) respects the requirement of initial and final slopes given by (20). The gain of the selected transfer function together with its piecewise-linear aproximation (i) are shown in next figures, for frequencies varying from 0 to 10. Outside this interval, both gain and phase of the transfer function do not exhibit important variations. The phase and the approximation are also shown (ii). If noise is present, then it can affect the quality of phase approximation. The gain (iii) and phase approximation (iv) using as test signal one that is affected by random perturbations are also ploted.

1. signal affected by a complex random perturbation

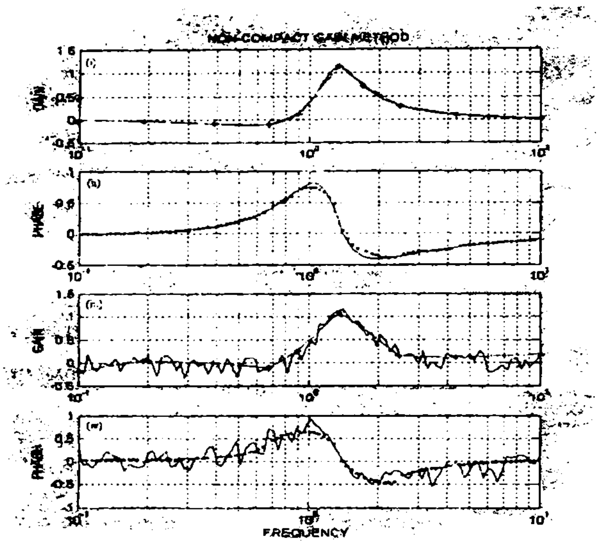

Fig. 4. Gain (-) and gain samples (*) (i) used in linear approximation of gain for (26); phase (-) and phase approximations (.) with this linear approximation of gain (ii); gain (-) and gain samples (*) (iii), respectively phase (-) and phase approximations (.) (iv) for signals affected by complex random perturbations

2. system function affected by a real random perturbation

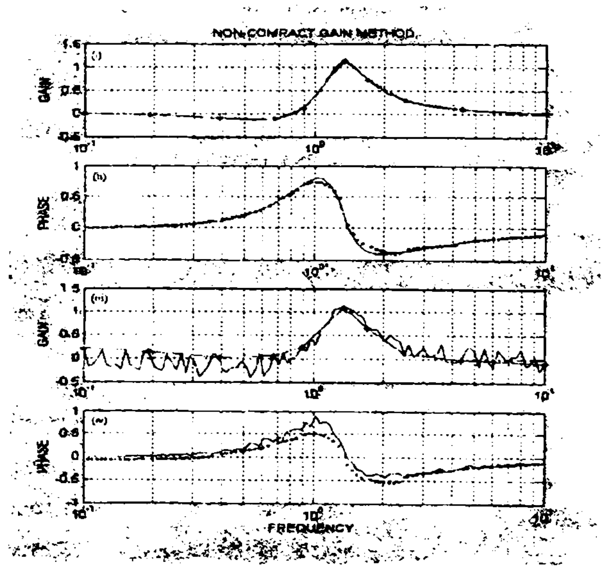

Fig. 5. Gain (-) and gain samples (*) (i) used in linear approximation of gain for (26); phase (-) and phase approximations ( ) with this linear approximation of gain (ii); gain (-) and gain samples (*) (iii), respectively phase (-) and phase approximations (.) (iv) for signals affected by real random perturbations

3. parameters of the modified Bode transfer function affected by a real random perturbation

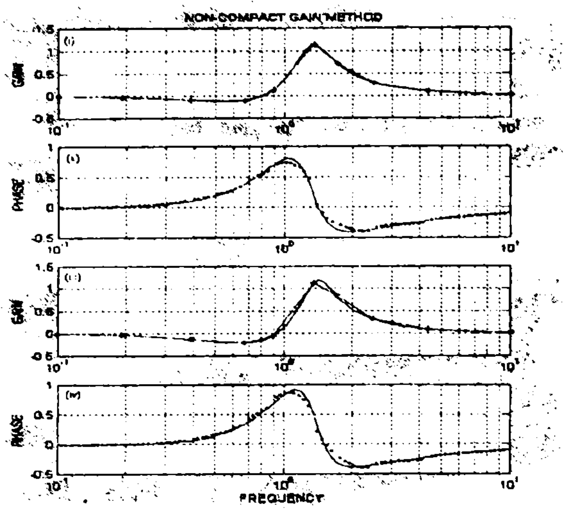

Fig 6 Gain (-) and gain samples (*) (i) used in linear approximation of gain for (26), phase (-) and phase approximations ( ) with this linear approximation of gain (ii), gain (-) and gain samples (*) (iii), respectively phase (-) and phase approximations ( ) (iv) for signals whose parameters are affected by real random perturbations

7. CONCLUSIONS

We have presented two methods for phase approximations: non-compact gain technique for linear frequency domain and the approach based on logarithmic sampling of gain for logarithmic frequency domain, using signals affected by random perturbations. A comparison of the behavior of these algorithms, considering signals affected by perturbations, respectively signals that are not affected by perturbations, was also presented.

Unlike the non-compact gain technique where we need only the gain samples, the logarithmic sampling of gain requests for two parameters: Δ which describes the frequency sampling, and k used to provide a satisfactory approximation of the integral in (7). There is a trade-off between Δ and k [7]. Indeed, as k increases, Δ has to tend to one more rapidly. For this reason we have considered $\Delta = 2^{m(q)}$, with $m(q) = 2^{2-q}$ for $q = 1,\ldots,9$. The gain samples for non-compact gain technique were available by sampling with Δ the frequency interval $\omega \in [0.01, 10]$.

From the experimental results we can conclude that best achievements can be obtained using the linear frequency domain approximation, unless when the number of gain samples is low, where the logarithmic sampling of gain is superior, but we can also see that both methods for phase approximation behave well when we have considered as test signals, signals affected by random perturbations.

REFERENCES

[1] H. W. Boue, Networn analysis and feedback amplifier design, D. Van Nostrand, Princeton, NJ, 1945
[2] B. D. O. Anderson and M. Green, "Hilbert transform and gain/phase error bounds for rational functions," IEEE Transactions on Circuits and Systems, vol 35, no 5, pp 528–535, May 1988
[3] M. Gr___ a_d B. D. O. A_derso_, "On the co_ti_uity of th_ Wiener-Hopf factorization operation," J. Austral Math Soc Ser B, vol. 28, pp 443–461, 1987
[.] _ C N_wt__, L __ G__ld, __d __ F K_is__, ____yti___ design of linear feedback controls, John Wiley & Sons, Inc., London, 1957
[5] C Rusu, P Kuosmanen, and A Burian, "1-D non-minimum phase retrieval by gain sampling," in Proc. ECCTD'99, Stresa, Italy, Sept 1999, vol 2, pp 755–758
[6] A. Papoulis, The Fourier integral and its applications, McGraw-Hill, 1962.
[7] Corneliu Rusu and Pauli Kuosmanen, "Phase approximation by logarithmic sampling of gain," IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing, vol 50, no 2, pp. 93–101, Feb. "003.
[8] Lacrimioara Buzan, "Phase approximation by gain samples," M.S. Thesis, Technical University of Cluj-Napoca, 2003
[9] Peter Henrici, Applied and Computational Complex Analysis, vol. I: Power Series - Integration - Conformal Mapping - Location of Zeros, John Wiley & Sons, New-York, 1974
[10] A. Papoulis, Signal analysis, McGraw-Hill, 1977
[11] Lacrimioara Buzan, "Phase Approximation in Linear and Logaritmic Frequency Domain", M.S. Thesis, Technical University of Cluj-Napoca, 2004
[12] Lacrimioara Buzan, Corneliu Rusu, Radu Ciprian Bilcu, Pauli Kuosmanen, "Phase Approximation in Linear and Logarithmic Frequency Domain", Proceedings of the First International Symposium on Control, Communications and Signal Processing, Hammamet, Tunisia, 21-24 March 2004, pp.709-712, ISBN: 0-7803-8380-X
[13] L. A. Zadeh and C. A. Desoer, *Linear System Theory*, McGraw-Hill, New York, 1969