SCIENTIFIC BULLETIN
OF
THE POLITEHNICA UNIVERSITY TIMISOARA

Vol 63(77), Issue 1 MATHEMATICS –PHYSICS 2018

# ON UNIFORM POLYNOMIAL DICHOTOMY IN BANACH SPACES

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**Abstract**

The main objective of the present paper is to describe the polynomial dichotomy behaviour in the uniform case of evolution operators in Banach spaces. In this sense we generalize the uniform polynomial stability notion by giving necessary and sufficient conditions for the dichotomy concept. [1]
Keywords and phrases: *evolution operator, uniform polynomial dichotomy*

## 1 Introduction

The concept of exponential dichotomy was introduced in 1930 by O. Perron [4] and it has been studied for many years. Even though nowadays it plays an important role in the theory of dynamical systems, there are some situations in which the notion of exponential dichotomy is too restrictive for the dynamics and for this reason it is important to have in mind a more general type of dichotomic behavior. In this sense, we refer to the polynomial dichotomy notion, which was firstly mentioned for the nonuniform case by Barreira and Valls in [1]. Moreover, the are many other works that deal with the polynomial asymptotic behaviors of evolution operators [2], [3], [5].
The aim of this paper is to give characterization theorems for the uniform polynomial dichotomy concept. The obtained results generalizes some well-known theorems given for the stability property.

## 2 Preliminaries

Let $X$ be a real or complex Banach space and $\mathrm{B}(X)$ the Banach algebra of all bounded linear operators acting on $X$. The norms on $X$ and on $\mathrm{B}(X)$ will be denoted by $\|.\|$ . The identity operator on $X$ is denoted by $I$. We also denote by

[1] MSC (2010): *34D05, 34D09*

$$\Delta = \{(t,s) \in R_+^2 : t \geq s\}, \qquad \Delta_1 = \{(t,s) \in \Delta : s \geq 1\}$$

and

$$T = \{(t,s,t_0) \in R_+^3 : t \geq s \geq t_0\}, \qquad T_1 = \{(t,s,t_0) \in T : t_0 \geq 1\}.$$

**Definition 2.1.** *An application $U : \Delta \to B(X)$ is said to be an evolution operator on $X$ if*

*$(e_1)$ $U(t,t) = I$ for every $t \geq 0$*

*$(e_2)$ $U(t,s)U(s,t_0) = U(t,t_0)$ for all $(t,s,t_0) \in T$.*

**Definition 2.2.** *An evolution operator $U : \Delta \to B(X)$ is said to be* strongly measurable *if for all $(s,x) \in R_+ \times X$, the mapping $t \mapsto \|U(t,s)x\|$ is measurable on $[s,\infty)$.*

**Definition 2.3.** *An application $P : R_+ \to B(X)$ is said to be a projection family on $X$ if $P^2(t) = P(t)$, for all $t \geq 0$.*

**Remark 2.1.** *If $P : R_+ \to B(X)$ is a projection family on $X$, then the mapping $Q : R_+ \to B(X), Q(t) = I - P(t)$ is also a projection family on $X$, which is called the complementary projection of $P$.*

**Definition 2.4.** *A projection family $P : R_+ \to B(X)$ is said to be invariant to the evolution operator $U : \Delta \to B(X)$ if*

$$U(t,s)P(s) = P(t)U(t,s),$$

*for all $(t,s) \in \Delta$.*

In what follows, if $P : R_+ \to \mathrm{B}(X)$ is an invariant projection family to the evolution operator $U : \Delta \to \mathrm{B}(X)$, we will say that $(U,P)$ is a dichotomic pair.

**Definition 2.5.** *The pair $(U,P)$ is* uniformly polynomially dichotomic (u.p.d.) *if there are $N \geq 1$ and $\nu > 0$ such that:*

*$(upd_1)$* $(t+1)^\nu \|U(t,s)P(s)x\| \leq N(s+1)^\nu \|P(s)x\|$

*$(upd_2)$* $(t+1)^\nu \|Q(s)x\| \leq N(s+1)^\nu \|U(t,s)Q(s)x\|$

*for all $(t,s,x) \in \Delta \times X$.*

**Definition 2.6.** *The pair $(U,P)$ is* uniformly logarithmic dichotomic (u.l.d.) *if there exists $L > 1$ such that:*

$(uld_1)$ $\|U(t,s)P(s)x\| \ln \dfrac{t+1}{s+1} \leq L\|P(s)x\|$

$(uld_2)$ $\|Q(s)x\| \ln \dfrac{t+1}{s+1} \leq L\|U(t,s)Q(s)x\|$

*for all* $(t,s,x) \in \Delta \times X$.

**Definition 2.7.** *The pair* $(U,P)$ *is* uniformly dichotomic (u.d.) *if there exists* $N \geq 1$ *such that*

$(ud_1)$ $\|U(t,s)P(s)x\| \leq N\|P(s)x\|$

$(ud_2)$ $\|Q(s)x\| \leq N\|U(t,s)Q(s)x\|$

*for all* $(t,s,x) \in \Delta \times X$.

**Definition 2.8.** *The pair* $(U,P)$ has uniform polynomial growth (u.p.g.) *if there are* $M \geq 1$ *and* $\omega > 0$ *such that*

$(upg_1)$ $(s+1)^\omega \|U(t,s)P(s)x\| \leq M(t+1)^\omega \|P(s)x\|$

$(upg_2)$ $(s+1)^\omega \|Q(s)x\| \leq M(t+1)^\omega \|U(t,s)Q(s)x\|$

*for all* $(t,s,x) \in \Delta \times X$.

**Remark 2.2.** *It is obvious that*

$$u.p.d. \Rightarrow u.d. \Rightarrow u.p.g.$$

# 3 Uniform polynomial dichotomy

**Lemma 3.1.** *Let* $U : \Delta \to B(X)$ *be an evolution operator and* $P : R_+ \to B(X)$ *a projection family invariant to* $U$*. If* $(U,P)$ *is u.l.d. then there exists* $L > 1$ *such that for all* $(t,s) \in \Delta_1$ *there exists* $n \in \mathbb{N}$ *with the following properties:*

**(i)** $se^{4nL} \leq t < se^{4(n+1)L}$

**(ii)** $\|U(se^{4nL},s)P(s)x\| \leq \dfrac{1}{2^n}\|P(s)x\|$

**(iii)** $\|U(se^{4nL},s)Q(s)x\| \geq 2^n\|Q(s)x\|, \ \forall x \in X$

*Proof.* It follows immediately by taking $n = \left[\ln\left(\dfrac{t}{s}\right)^{\frac{1}{4L}}\right]$.

□

The next theorem is a logarithmic criterion for the uniform polynomial dichotomy concept.

**Theorem 3.2.** *The pair $(U,P)$ is uniformly polynomially dichotomic if and only if $(U,P)$ has uniform polynomial growth and $(U,P)$ is uniformly logarithmic dichotomic.*

*Proof. Necessity.* We suppose that $(U,P)$ is u.p.d. Then, from Remark 2.2 we obtain that $(U,P)$ has u.p.g. We prove that $(U,P)$ is u.l.d. We consider the application

$$f:[1,\infty)\to R, f(t)=\frac{\ln t}{t},$$

with $f(t)\le\frac{1}{e}$. Then, for the first condition $(uld_1)$ we have

$$\|U(t,s)P(s)x\|\ln\frac{t+1}{s+1}\le N\left(\frac{s+1}{t+1}\right)^{\nu}\|P(s)x\|\ln\frac{t+1}{s+1}=$$
$$\frac{N}{\nu}\left(\frac{s+1}{t+1}\right)^{\nu}\|P(s)x\|\ln\left(\frac{t+1}{s+1}\right)^{\nu}=$$
$$\frac{N}{\nu}\|P(s)x\|f\left(\left(\frac{t+1}{s+1}\right)^{\nu}\right)\le$$

$$\le\frac{N}{\nu}\cdot f(t)\|P(s)x\|\le\frac{N}{\nu e}\|P(s)x\|.$$

For $(uld_2)$ we do in a similar manner and we obtain

$$\|Q(s)x\|\ln\frac{t+1}{s+1}\le\frac{N}{\nu e}\|U(t,s)Q(s)x\|.$$

So, we have that $(U,P)$ is u.l.d. for $L=\frac{N}{\nu e}+1$.

*Sufficiency.* Let $N=2Me^{4L\omega}$ and $\nu=\frac{\ln 2}{4L}$.

$$\|U(t,s)P(s)x\|=\|U(t,se^{4nL})U(se^{4nL},s)P(s)x\|\le$$

$$\le M\left(\frac{t+1}{se^{4nL}+1}\right)^{\omega}\|U(se^{4nL}+1,s)P(s)x\|\le$$

$$\le M\cdot e^{4L\omega}\cdot\frac{1}{2^n}\|P(s)x\|=\frac{N}{2^{n+1}}\|P(s)x\|=\frac{N}{e^{(n+1)\ln 2}}\|P(s)x\|\le$$

$$\le\frac{N}{\left(\frac{t+1}{s+1}\right)^{\frac{\ln 2}{4L}}}\|P(s)x\|=N\cdot\left(\frac{s+1}{t+1}\right)^{\nu}\|P(s)x\|$$

It results in the same way as $(upd_1)$.

□

Another characterization of uniform polynomial dichotomy concept is given by

**Theorem 3.3.** *The pair* $(U,P)$ *is uniformly polynomially dichotomic if and only if* $(U,P)$ *has uniform polynomial growth and there exists* $r>1$ *such that*

$(upH_1)$ $2\|U(rs,s)P(s)x\| \leq \|P(s)x\|$

$(upH_2)$ $\|U(rs,s)Q(s)x\| \geq 2\|Q(s)x\|$

*for all* $s \geq 1, x \in X$.

*Proof. Necessity* We suppose that $(U,P)$ is u.p.d. Then, from Remark 2.2 we obtain that $(U,P)$ has u.p.g. Now, let $r = 2(2N)^{\frac{1}{\nu}}$.

$(upH_1)$

$$\|U(rs,s)P(s)x\| \leq N\left(\frac{s+1}{rs+1}\right)^{\nu}\|P(s)x\| \leq N\cdot\left(\frac{2}{r}\right)^{\nu}\|P(s)x\|\frac{1}{2}\|P(s)x\|.$$

$(upH_2)$

$$\|U(rs,s)Q(s)x\| \geq \frac{\|Q(s)x\|}{N}\left(\frac{rs+1}{s+1}\right)^{\nu} \geq \frac{\|Q(s)x\|}{N}\cdot\left(\frac{r}{2}\right)^{\nu} = 2\|Q(s)x\|.$$

*Sufficiency* Let $(t,s)\in\Delta_1$ and $n = \left[\ln\left(\frac{t}{s}\right)^{\frac{1}{\ln r}}\right]$. Then we obtain the relation $sr^n \leq t < sr^{n+1}$. In order to prove that $(U,P)$ is u.p.d., we show that $(U,P)$ is u.l.d. and then we use Theorem 3.2.

$(upL_1)$

$$\begin{aligned}
&\|U(t,s)P(s)x\| = \|U(t,sr^n)P(sr^n)U(sr^n,s)P(s)x\| \leq \\
&\leq M\left(\frac{t+1}{sr^n+1}\right)^{\omega}\|P(sr^n)U(sr^n,s)P(s)x\| \leq \\
&\leq M\cdot(r+1)^{\omega}\|U(sr^n,s)P(s)x\| = \\
&= M(r+1)^{\omega}\|U(sr^n,sr^{n-1})P(sr^{n-1})U(sr^{n-1},s)P(s)x\| \leq \\
&\leq \frac{M}{2}(r+1)^{\omega}\|U(sr^{n-1},s)P(s)x\| \leq \cdots \leq \frac{2M(r+1)^{\omega}}{2^{n+1}}\|P(s)x\| \leq \\
&\leq \frac{\ln r}{\ln\frac{t+1}{s+1}}\cdot 2M(r+1)^{\omega}\|P(s)x\|
\end{aligned}$$

$(upL_2)$ We apply the evolution property and we use the same technique as in the previous case. We obtain

$$\|U(t,s)Q(s)x\| \geq \frac{\|Q(s)x\|\cdot\dfrac{1}{\ln r}\cdot\ln\dfrac{t+1}{s+1}}{2M(r+1)^{\omega}}$$

Finally, we have that $(U,P)$ is u.l.d. for $L = 2M(r+1)^{\omega}\ln r + 1$ and from Theorem 3.2, it results that $(U,P)$ is u.p.d. □

**Remark 3.4.** *The previous theorem is a generalization of some results proved by Hai in [3].*

In what follows, we will present a characterization of Datko type of the uniform polynomial dichotomy concept.

**Theorem 3.5.** *Let $(U,P)$ be a strongly measurable dichotomic pair with uniform polynomial growth. Then $(U,P)$ is uniformly polynomially dichotomic if and only if there exists $D > 1$ with*

$(upD_1)$ $$\int_t^{\infty}\frac{\|U(\tau,t_0)P(t_0)x_0\|}{\tau+1}d\tau \leq D\|U(s,t_0)P(t_0)x_0\|$$

$(upD_2)$ $$\int_{t_0}^{t}\frac{\|U(s,t_0)Q(t_0)x_0\|}{s+1}ds \leq D\|U(t,t_0)Q(t_0)x_0\|$$

*for all $(t,t_0,x_0)\in\Delta\times X$.*

*Proof. Necessity.* A simple computation shows that the relations $(upD_1)$ and $(upD_2)$ take place for $D = 1 + \frac{N}{\nu}$.

*Sufficiency.* Step 1. We show that $(U, P)$ is uniformly dichotomic.

$(ud_1)$ If $t \geq 2s + 1$ then

$$\|U(t,t_0)P(t_0)x_0\| = \frac{2}{t+1}\int_{\frac{t-1}{2}}^{t} \|U(t,t_0)P(t_0)x_0\| d\tau \leq$$

$$\leq 2M \int_{\frac{t-1}{2}}^{t} \left(\frac{t+1}{\tau+1}\right)^{\omega} \frac{\|U(\tau,t_0)P(t_0)x_0\|}{\tau+1} d\tau \leq$$

$$\leq DM2^{\omega}\|U(s,t_0)P(t_0)x_0\| = M_1\|U(s,t_0)P(t_0)x_0\|$$

where $M_1 = MD2^{\omega} + 1$.

If $t \in [s, 2s+1)$ then $\frac{t+1}{s+1} \leq 2$. We obtain

$$\|U(t,t_0)P(t_0)x_0\| \leq M \left(\frac{t+1}{s+1}\right)^{\omega} \|U(s,t_0)P(t_0)x_0\| \leq$$

$$\leq 2^{\omega} M\|U(s,t_0)P(t_0)x_0\| \leq M_1\|U(s,t_0)P(t_0)x_0\|.$$

$(ud_2)$ Analogous with $(ud_1)$.

Step 2. We prove that $U$ is u.p.d.

$(upl_1)$

$$\|U(t,t_0)P(t_0)x_0\| \ln \frac{t+1}{s+1} = \int_{s}^{t} \frac{\|U(t,t_0)P(t_0)x_0\|}{\tau+1} d\tau \leq$$

$$\leq M_1 \int_{s}^{t} \frac{\|P(\tau)U(\tau,t_0)x_0\|}{\tau+1} d\tau \leq \; DM_1\|U(s,t_0)P(t_0)x_0\|.$$

$(upl_2)$

$$\|Q(t_0)x_0\| \ln \frac{t+1}{s+1} = \int_{s}^{t} \frac{\|Q(t_0)x_0\|}{\tau+1} d\tau \leq M_1 \int_{s}^{t} \frac{\|U(\tau,t_0)Q(t_0)x_0\|}{\tau+1} d\tau \leq$$

$$\leq DM_1\|U(t,t_0)Q(t_0)x_0\|.$$

For $t_0 = s,\ x_0 = x$ and from Theorem (3.2) we obtain the conclusion.

□

The next theorem is a characterization which uses Lyapunov functions for the uniform polynomial dichotomy of an evolution operator.

**Theorem 3.6.** *Let $(U, P)$ be a strongly measurable dichotomic pair with uniform polynomial growth. Then $(U, P)$ is uniformly polynomially dichotomic if and only if there are $D > 1$ and $L : \Delta \times X \to R_+$ with the properties:*

**(*i*)** $L(t, t_0, x_0) \leq D\left(\|U(t, t_0)P(t_0)x_0\| + \|U(t, t_0)Q(t_0)x_0\|\right)$
*for all* $(t, t_0, x_0) \in \Delta \times X$

**(*ii*)** $L(t, t_0, P(t_0)x_0) + \int_s^t \frac{\|U(\tau, t_0)P(t_0)x_0\|}{\tau + 1} d\tau = L(s, t_0, P(t_0)x_0)$
*for all* $(t, s, t_0, x_0) \in T \times X$

**(*iii*)** $L(s, t_0, Q(t_0)x_0) + \int_s^t \frac{\|U(\tau, t_0)Q(t_0)x_0\|}{\tau + 1} d\tau = L(t, t_0, Q(t_0)x_0),$
*for all* $(t, s, t_0, x_0) \in T \times X.$

*Proof. Necessity.* If $U$ is u.p.d. then by Theorem (3.5) the function

$$L : \Delta \times X \to R_+$$

defined by

$$L(t, t_0, x_0) = \int_s^\infty \frac{\|U(\tau, t_0)P(t_0)x_0\|}{\tau + 1} d\tau + \int_{t_0}^t \frac{\|U(\tau, t_0)Q(t_0)x_0\|}{\tau + 1} d\tau$$

satisfies the conditions $(i) - (iii)$.

*Sufficiency.*
It follows from Theorem (3.5). □

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