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SOLVING FRACTIONAL ORDINARY DIFFERENTIAL EQUATION USING PLSM

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Abstract

In this paper, we obtaining analytical approximate solutions for fractional ordinary differential equations using *Polynomial Least Square Method* (*PLSM*). An example is illustrated to show the presented methods efficiency and convenience. [1]

Keywords and phrases: *Fractional ordinary differential equations, Polynomial Least Square Method(PLSM), Caputos fractional derivative*

1 Introduction

In recent years, fractional ordinary differential equations have been investigated by many authors. Fractional ordinary differential equations are generally used in many branches of science such as: mathematics, physics, chemistry and engineering.

Since most of these equations have no exact solutions, it has been necessary to develop numerical methods or analytical methods to find the approximate solutions of these equations.

In order to find approximate solutions of these equations, many methods were proposed, such as:

- Fractional Adams-Bashforth-Moulton method [2];
- Adomian decomposition method [4];
- Homotopy analysis method [3], [8];
- Variational iteration method [9], [10].

We consider the following fractional ordinary differential equation:

$$D^{\alpha}y(x) = f(x, y(x)) \tag{1}$$

[1]MSC (2010): 60H20, 34F15

$\alpha > 0$, with the initial condition:

$$y(0) = \nu_0 \tag{2}$$

where ν_0 are real constant and D^α denote the Caputo's fractional derivative:

$$D^\alpha \tilde{y}(x) = \frac{1}{\Gamma(n-\alpha)} \cdot \int_0^x (x-\zeta)^{n-\alpha-1} \cdot \tilde{y}^{(n)}(\zeta) d\zeta$$

$n-1 < \alpha < n$ where $n \in \mathbb{N}^*$.

In the next section we will introduce the *Polynomial Least Square Method* (*PLSM*) which allows us to determine analytical approximate polynomial solutions for fractional ordinary differential equations and in the third section we will compare our approximate solutions with approximate solutions presented by *fractional Adams-Bashforth-Moulton method* (*FABMM*).

2 The Polynomial Least Squares Method

We denote by $\tilde{y}$ an approximate solution of equation (1). The error obtained by replacing the exact solution y with the approximation $\tilde{y}$ is given by the remainder:

$$\mathcal{R}(x, \tilde{y}(x)) = D^\alpha \tilde{y}(x) - f(x, \tilde{y}(x)). \tag{3}$$

For $\epsilon \in \mathbb{R}_+$, we will compute approximate polynomial solutions $\tilde{y}$ of the problem $(1, 2)$ on the interval $[0, b]$.

Definition 2.1. *We call an* ϵ-approximate polynomial solution *of the problem* $(1, 2)$ *an approximate polynomial solution* $\tilde{y}$ *satisfying the relations*

$$|\mathcal{R}(\tilde{y})| < \epsilon \tag{4}$$

$$\tilde{y}(0) = \nu_0. \tag{5}$$

We call a *weak* ϵ*-approximate polynomial solution* of the problem $(1, 2)$ an approximate polynomial solution $\tilde{y}$ satisfying the relation:

$$\int_0^b |\mathcal{R}(\tilde{y})| dx \leq \epsilon \tag{6}$$

together with the initial conditions (5).

Definition 2.2. *Let* $P_m(x) = c_0 + c_1x + c_2x^2 + \cdots + c_mx^m,\ c_i \in \mathbb{R},\ i = \overline{0, m}$ *be a sequence of polynomials satisfying the condition:*

$$P_m(0) = \nu_0.$$

We call the sequence of polynomials $P_m(x)$ *convergent to the solution of the problem* $(1, 2)$ *if* $\lim\limits_{m\to\infty} D(P_m(x)) = 0$.

We observe that from the hypothesis of the initial problems $(1, 2)$ it follows that there exists a sequence of polynomials $P_m(x)$ which converges to the solution of the problem.

We will compute a weak ϵ - approximate polynomial solution, in the sense of the Definition 2.1, of the type:

$$\tilde{y}(x) = \sum_{k=0}^{m} d_k x^k \tag{7}$$

where $d_0, d_1, \cdots, d_m$ are constants which are calculated using the following steps:

- By substituting the approximate solution (7) in the equation (1) we obtain the expression:

  $$\mathcal{R}(\tilde{y}) = D^\alpha \tilde{y}(x) - f(x, \tilde{y}(x)). \tag{8}$$

  If we could find $d_0, d_1, \cdots, d_m$ such $\mathcal{R}(\tilde{y}) = 0,\ \tilde{y}(0) = \nu_0$, then by substituting $d_0, d_1, \cdots, d_m$ in (7) we obtain the solutions of equation (1).

- Then we attach to the problem (1,2) the following functional:

  $$\mathcal{J}(d_1, d_2, d_3, \cdots, d_m) = \int_0^b \mathcal{R}^2(\tilde{y})dx \tag{9}$$

  where d_0 is computed as functions of $d_1, d_2, d_3, \cdots, d_m$ using the initial condition (5).

- We compute the values $d_1^0, d_2^0, d_3^0, \cdots, d_m^0$ as the values which give the minimum of the functional $\mathcal{J}$, and the values of d_0 is function of $d_1^0, d_2^0, d_3^0, \cdots, d_m^0$ using the initial condition.

- With constants $d_1^0, d_2^0, d_3^0, \cdots, d_m^0$ previously determined we consider the polynomial:

  $$M_m(x) = \sum_{k=0}^{m} d_k^0 x^k. \tag{10}$$

Theorem 2.1. *The sequence of polynomials $M_m(x)$ from (10) satisfies the property:*

$$\lim_{x\to\infty}\int_0^b \mathcal{R}^2(M_m(x))dx = 0. \qquad (11)$$

Moreover, $\forall\epsilon > 0,\ \exists m_o \in \mathbb{N},\ m > m_0$ *it follows that* $M_m(x)$ *is a* weak ϵ-approximate polynomial solution *of the problem* $(1, 2)$.

Proof. Based on the way the polynomials $M_m(x)$ are computed and taking into account the relations (8)-(11), the following inequalities are satisfied:

$$0 \le \int_0^b \mathcal{R}^2(M_m(x))dx \le \int_0^b \mathcal{R}^2(P_m(x))dx,\ \forall m \in \mathbb{N},$$

where $P_m(x)$ is the sequence of polynomials introduced in Definition 2.2.

It follows that:

$$0 \le \lim_{x\to\infty}\int_0^b \mathcal{R}^2(M_m(x))dx \le \lim_{x\to\infty}\int_0^b \mathcal{R}^2(P_m(x))dx = 0.$$

We obtain:

$$\lim_{x\to\infty}\int_0^b \mathcal{R}^2(M_m(x))dx = 0.$$

From this limit we obtain that $\forall\epsilon > 0,\ \exists m_o \in \mathbb{N},\ m > m_0$ it follows that $M_m(x)$ is a *weak ϵ-approximate polynomial solution* of the problem $(1, 2)$. □

In order to find ϵ-approximate polynomial solutions of the problem (1,2) by using the Polynomial Least Squares Method we will first determine weak approximate polynomial solutions, $\tilde{y}$.

If $|\mathcal{R}(\tilde{y})| < \epsilon$ then $\tilde{y}$ is also an ϵ approximate polynomial solution of the problem.

3 Application

We consider the following linear fractional differential equation ([2]):

$$D^\alpha y(x) + y(x) - x^{\alpha+3} - \frac{\Gamma(4+\alpha)}{6}\cdot x^3 = 0 \qquad (12)$$

$\alpha = 0,25$; $x \in [0, \frac{1}{30}]$ and the initial condition: $y(0) = 0$.

The exact solution of the problem is:

$$y(x) = x^{3+\alpha}.$$

A numerical solutions for this problem is presented by Baskonus at all in [2] using *fractional Adams-Bashfort-Moulton method* (*FABMM*).

Using (*PLSM*):

- We compute a solution of the type:

$$\tilde{y}(x) = d_0 + d_1 \cdot x^1 + d_2 \cdot x^2 + d_3 \cdot x^3 + d_4 \cdot x^4$$

  with initial condition: $\tilde{y}(0) = 0$ we obtain: $d_0 = 0$.

- The approximate solution becomes:

$$\tilde{y}(x) = d_1 \cdot x^1 + d_2 \cdot x^2 + d_3 \cdot x^3 + d_4 \cdot x^4.$$

- The corresponding remainder is:

$$\mathcal{R}(x) = \frac{4x^{3/4}\left(385d_1 + 8x\left(55d_2 + 60d_3x + 64d_4x^2\right)\right)}{1155\Gamma\left(\frac{3}{4}\right)} + \\ + d_1x + d_2x^2 + d_3x^3 + d_4x^4 - x^{13/4} - \frac{1}{6}x^3\Gamma\left(\frac{17}{4}\right). \quad (13)$$

  Next we compute:

$$\mathcal{J}(d_1, d_2, d_3, \cdots, d_m) = \int_0^{\frac{1}{30}} \mathcal{R}^2(\tilde{y})dx$$

  and minimize it obtaining the values:

$$d_1 = 3,53901 \cdot 10^{-6};\ d_2 = 0,00131029;\ d_3 = 0,387136,\ d_4 = 2,29079.$$

- The approximate analytical solution of the problem (12) using (*PLSM*) is:

$$\tilde{y}(x) = 3,53901 \cdot 10^{-6} \cdot x + 0,00131029 \cdot x^2 + 0,387136 \cdot x^3 + 2,29079 \cdot x^4.$$

Table 1 present the comparison between absolute errors coresponding to the numerical solution proposed by Baskonus in [2] using (*FABMM*) and aur solution (*PLSM*).

From the table, it is easy to see that using (*PLSM*) results are better than using (*FABMM*).

Additionally, (*PLSM*) obtains the analytical solution of the polynomial form of the problem, not only numerical solutions, thus demonstrating the usefulness and accuracy of the (*PLSM*).

Table 1: Numerical results

x	$Exactsolution$	$Error(FABMM)$	$Error(PLSM)$
0.0033333	2.82×10^{-3}	3.8343×10^{-9}	2.9598×10^{-9}
0.0066667	1.73×10^{-3}	2.1194×10^{-8}	7.4355×10^{-11}
0.0100000	3.31×10^{-4}	5.4419×10^{-8}	1.8279×10^{-9}
0.0133333	1.15×10^{-3}	1.0405×10^{-7}	1.1658×10^{-9}
0.0166667	1.75×10^{-3}	1.7047×10^{-7}	6.2667×10^{-10}
0.0200000	2.36×10^{-3}	2.5705×10^{-7}	1.8004×10^{-9}
0.0233333	1.49×10^{-3}	3.5512×10^{-7}	1.2389×10^{-9}
0.0266667	2.66×10^{-3}	4.7380×10^{-7}	7.2161×10^{-10}
0.0300000	4.88×10^{-3}	6.1050×10^{-7}	1.7042×10^{-9}
0.0333333	0	7.6535×10^{-7}	3.1652×10^{-9}

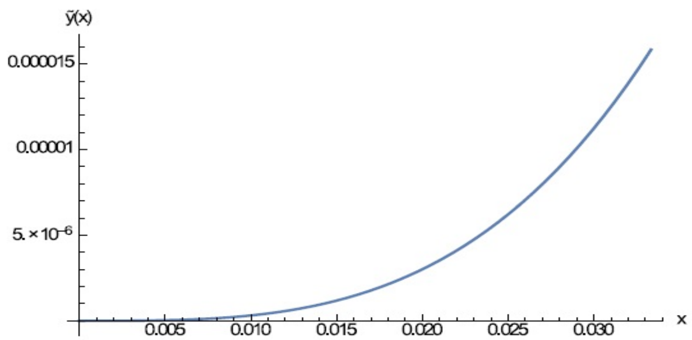

Figure 1 - The approximate analytical solution using (*PLSM)*

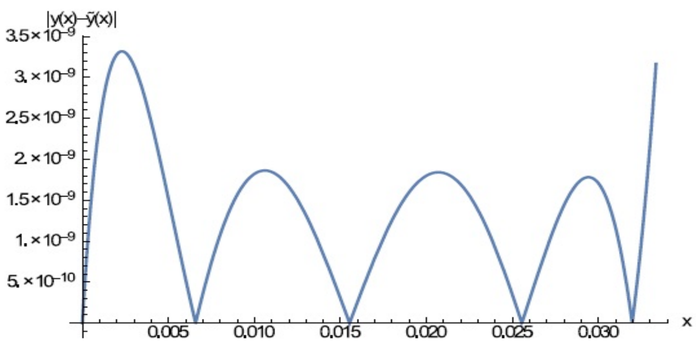

Figure 2 - The absolute errors corresponding to the approximations given by (*PLSM)*

4 Conclusions

The computations performed show that (*PLSM*) allows us to obtain approximations with an error relative to the exact or numerical solution smaller than the errors obtained using by *fractional Adams-Bashforth-Moulton method* (*FABMM*).

The application presented emphasize the high accuracy of the method by means of a comparison with previous results.

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