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# STABILITY PROBLEMS AND NUMERICAL INTEGRATION ON THE POINCARÉ LIE GROUP

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**Abstract**

An underactued drift-free left-invariant control system on the Lie group $ISO(3,1)$ is analyzed. [1]

Keywords and phrases: *spectral stability, Lie group*

## 1 Introduction

The Poincaré group $ISO(3,1)$ was first defined by Minkowski (1908) as the group of Minkowski space-time isometries. It can be written as a semi-direct product of the Lorentz group SO(3,1) with the four-dimensional translation group $R^4$. Due to its big importance in quantum theory of fields, we are interested to study an optimal control problem on this Lie group. The interest in such problems arise from their deep applications in engineering (spacecraft dynamics, sub-aquatic dynamics, the tower control problem), in chemistry (molecular motion control) or physics (quantum theory).

## 2 An optimal control problem on the Poincaré Lie group

Let us consider $\{J_i, K_i, P_i, H\}_{(1=i,j=3)}$ the usual generators of spatial rotations, boosts, space translations, and time translation respectively, of the Poincaré inhomogeneous Lie algebra $iso(3,1)$; the nonzero brackets are given by:

$$[J_i, J_j] = \varepsilon_{ijk} J_k;\ [J_i, P_j] = \varepsilon_{ijk} P_k;\ [J_i, K_j] = \varepsilon_{ijk} K_k;$$

$$[H, K_j] = P_i;\ [K_i, K_j] = -\varepsilon_{ijk} J_k;\ [P_i, K_i] = H.$$

[1]MSC(2010): 34A26, 34H05, 34M45, 35A24, 37C10, 49J15, 49K15, 93C15

A general left invariant drift free control system on the Poincaré Lie algebra $iso(3,1)$ with fewer controls than state variables can be written in the following form:

$$\dot{\mathrm{X}} = X(\sum_{i=1}^{m} u_i A_i),$$

where $X \in ISO(3,1)$, the functions $u_i$ are the control inputs, and $m < 10$. In all that follows, we shall concentrate to the following left-invariant, drift-free control system on $ISO(3,1)$ with 4 controls:

$$\dot{\mathrm{X}} = X(u_1 J_1 + u_2 K_1 u_3 K_2 + u_4 H). \qquad (1)$$

**Theorem 2.1** *The system (1) is controllable.*

**Proof:** Since the span of the set of Lie brackets generated by $J_1, K_1, K_2, H$ coincides with $iso(3,1)$, the Proposition is a consequence of a result due to Jurdjevic and Sussman, see [6].

Let $C$ be the cost function given by:

$$C(u_1, u_2, u_3, u_4) = \frac{1}{2}\int_0^{t_f} [u_1^2(t) + u_2^2(t) + u_3^2(t) + u_4^2(t)]dt.$$

The controls that minimize $C$ and steer the system (1) from the initial state $X = X_0$ at $t = 0$ to the final state $X = X_f$ at $t = t_f$ are giving by the solutions of the following differential equations:

$$\begin{cases} j_1^{'} = k_2 k_3 \\ j_2^{'} = -j_1 j_3 - k_1 k_3 \\ j_3^{'} = j_1 j_2 \\ k_1^{'} = -k_2 j_3 + h p_1 \\ k_2^{'} = -k_3 j_1 + k_1 j_3 + h p_2 \\ k_3^{'} = 2 j_1 k_2 - j_2 k_1 + h p_3 \\ p_1^{'} = -h k_1 \\ p_2^{'} = -j_1 p_3 - h k_2 \\ p_3^{'} = j_1 p_2 \\ h^{'} = k_1 p_1 - p_2 k_2 \end{cases} \qquad (2)$$

The system is obtained by applying Krishnaprasad's theorem (see [7]) to the optimal Hamiltonian given by:

$$H_{opt} = \frac{1}{2}(j_1^2 + k_1^2 + k_2^2 + h^2).$$

**Theorem 2.2** *The dynamics (2) has the following Poisson realization:*

$$(iso(3,1), \Pi_-, H),$$

*where:*

$$\Pi_- = \begin{pmatrix} 0 & j_3 & -j_2 & 0 & k_3 & -k_2 & 0 & p_3 & -p_2 & 0 \\ -j_3 & 0 & j_1 & -k_3 & 0 & k_1 & -p_3 & 0 & p_1 & 0 \\ j_2 & -j_1 & 0 & k_2 & -k_1 & 0 & p_2 & -p_1 & 0 & 0 \\ 0 & k_3 & -k_2 & 0 & -j_3 & j_2 & h & 0 & 0 & p_1 \\ -k_3 & 0 & k_1 & j_3 & 0 & -j_1 & 0 & h & 0 & p_2 \\ j_2 & -k_1 & 0 & -j_2 & j_1 & 0 & 0 & 0 & h & p_3 \\ 0 & p_3 & -p_2 & -h & 0 & 0 & 0 & 0 & 0 & 0 \\ -p_3 & 0 & p_1 & 0 & -h & 0 & 0 & 0 & 0 & 0 \\ p_2 & -p_1 & 0 & 0 & 0 & -h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -p_1 & -p_2 & -p_3 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (3)$$

*is the minus-Lie-Poisson structure on* $iso(3,1)$, *and*

$$H = \frac{1}{2}(j_1^2 + k_1^2 + k_2^2 + h^2)$$

*is the Hamiltonian function.*

**Proof:** Indeed, it is not hard to see that the dynamics (2) can be written as

$$\begin{pmatrix} j_1^{'} & j_2^{'} & j_3^{'} & k_1^{'} & k_2^{'} & k_3^{'} & p_1^{'} & p_2^{'} & p_3^{'} & h^{'} \end{pmatrix}^t = \Pi_- \cdot H$$

and $\Pi_-$ is the minus-Lie-Poisson structure on $iso(3,1)$.

**Corollary 2.1** *The Lie-Poisson structure* $\Pi_-$ *admits two linear independent Casimir operators:*

$$C_1 = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2 - h^2), \quad (4)$$

*and*

$$C_2 = (-hj_3 - p_1k_2 + p_2k_1)^2 + (hk_2 - p_1k_3 + p_3k_1)^2 +$$

$$+(-hj_1 - p_2k_3 + p_3k_2)^2 - (p_3j_3 + p_1j_1 + p_2j_2)^2. \quad (5)$$

## 3 Stability

The goal of this section is to analyze the spectral stability of the equilibrium states of the dynamics (2):

$$e_1^{MNPQRS}=(0,M,N,0,0,P,Q,R,S,0), e_2^{MNPQ}=(0,0,0,M,N,0,P,-\frac{MP}{N},Q,0),$$

$$e_3^{MNPQ}=(0,M,0,0,N,0,P,0,Q,0),\quad e_4^{MNPQ}=(0,0,M,0,0,N,P,0,Q,0),$$

$$e_5^{MNP}=(0,0,0,M,0,0,0,N,P,0),\quad e_6^{MN}=(M,0,0,N,0,0,0,0,0,0,),$$

$$e_7^{MN}=(M,0,0,0,0,0,N,0,0,0),\quad e_8^{MN}=(-\frac{M}{\sqrt{2}},0,0,0,N,0,0,0,\sqrt{2},M),$$

$$e_9^{MN}=(M,0,0,0,0,0,0,0,0,N),\quad e_{10}^{MNPQ}=(0,M,N,0,0,P,0,0,0,Q).$$

**Theorem 3.1** *(i) The equilibrium states $e_1^{MNPQRS}$ are spectrally stable iff $P\neq 0$ and $Q\neq 0$.*

*(ii) The equilibrium states $e_2^{MNPQ}$ are unstable for any nonzero reals $M,N,P,Q$.*

*(iii) The equilibrium states $e_3^{MNPQ}$ are spectrally stable iff $N=0$.*

*(iv) The equilibrium states $e_4^{MNPQ}$ are spectrally stable iff $P\neq 0$ and $N\neq 0$.*

*(v) The equilibrium states $e_5^{MNP}$ are are unstable for any nonzero reals $M,N,P$.*

*(vi) The equilibrium states $e_6^{MN}$ are are unstable for any nonzero reals $M,N$.*

*(vii) The equilibrium states $e_7^{MN}$ are are spectrally stable for any reals $M,N$.*

*(viii) The equilibrium states $e_8^{MN}$ are are unstable for any nonzero reals $M,N$.*

*(ix) The equilibrium states $e_9^{MN}$ are spectrally stable iff $M<-\frac{|N|}{\sqrt{2}}$ or $M>\frac{|N|}{\sqrt{2}}$.*

*(x) The equilibrium states $e_{10}^{MNPQ}$ are spectrally stable iff $Q\neq 0$.*

**Proof:** Let $A$ be the matrix of the linear part of the system (2):

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & k_3 & k_2 & 0 & 0 & 0 & 0 \\ -j_3 & 0 & -j_1 & -k_3 & 0 & -k_1 & 0 & 0 & 0 & 0 \\ j_2 & j_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k_2 & 0 & -j_3 & 0 & h & 0 & 0 & p_1 \\ -k_3 & 0 & k_1 & j_3 & 0 & -j_1 & 0 & h & 0 & p_2 \\ 2k_2 & -k_1 & 0 & -j_2 & 2j_1 & 0 & 0 & 0 & h & p_3 \\ 0 & 0 & 0 & -h & 0 & 0 & 0 & 0 & 0 & -k_1 \\ -p_3 & 0 & 0 & 0 & -h & 0 & 0 & 0 & -j_1 & -k_2 \\ p_2 & 0 & 0 & 0 & 0 & 0 & 0 & j_1 & 0 & 0 \\ 0 & 0 & 0 & -p_1 & -p_2 & 0 & -k_1 & -k_2 & 0 & 0 \end{pmatrix}.$$

The corresponding eigenvalues of the linearized $A(e_1)$ are $\lambda_i = 0, i = \overline{1,6}$, and

$$\lambda_{7,8,9,10} = \pm\sqrt{\frac{-N^2 - P^2 - Q^2 - R^2 \pm \sqrt{-4P^2Q^2 + (N^2 + P^2 + Q^2 + R^2)^2}}{2}},$$

so the assertion follows immediately.

Similar arguments provides us all the statements.

# 4 Numerical Integration via Lie-Trotter Integrator

We shall discuss now the numerical integration of the dynamics (2) via the Lie-Trotter integrator (see [11]). For the beginning, let us observe that the Hamiltonian vector field $X_H$ splits as follows:

$$X_H = X_{H_1} + X_{H_2} + X_{H_3} + X_{H_4},$$

where

$$H_1 = \frac{1}{2}j_1^2,\ H_2 = \frac{1}{2}k_1^2,\ H_3 = \frac{1}{2}k_2^2,\ H_4 = \frac{1}{2}h^2.$$

Their corresponding integral curves are respectively given by:

$$\begin{bmatrix} j_1(t) \\ j_2(t) \\ j_3(t) \\ k_1(t) \\ k_2(t) \\ k_3(t) \\ p_1(t) \\ p_2(t) \\ p_3(t) \\ h(t) \end{bmatrix} = A_i \begin{bmatrix} j_1(0) \\ j_2(0) \\ j_3(0) \\ k_1(0) \\ k_2(0) \\ k_3(0) \\ p_1(0) \\ p_2(0) \\ p_3(0) \\ h(0) \end{bmatrix},\ i = 1, 2, 3, 4,$$

where

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & e^{-at}-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{at}-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & e^{-at}-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{at}-1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & e^{-at}-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{at}-1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad a = j_1(0),$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & e^{-bt}-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & e^{bt}-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{bt}-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-bt}-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & e^{-bt}-1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-bt}-1 & 0 & 0 & 1 \end{bmatrix}, \quad b = k_1(0),$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & e^{ct}-1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & e^{-ct}-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{ct}-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ e^{ct}-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & e^{-ct}-1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{-ct}-1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad c = k_2(0),$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{dt} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{dt} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{dt} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad d = h(0).$$

Then, the Lie-Trotter integrator is given by:

$$\begin{bmatrix} j_1^{n+1} \\ j_2^{n+1} \\ j_3^{n+1} \\ k_1^{n+1} \\ k_2^{n+1} \\ k_3^{n+1} \\ p_1^{n+1} \\ p_2^{n+1} \\ p_3^{n+1} \\ h^{n+1} \end{bmatrix} = A_1A_2A_3A_4 \begin{bmatrix} j_1^{n} \\ j_2^{n} \\ j_3^{n} \\ k_1^{n} \\ k_2^{n} \\ k_3^{n} \\ p_1^{n} \\ p_2^{n} \\ p_3^{n} \\ h^{n} \end{bmatrix}, \qquad (6)$$

i.e.

$$\begin{cases} j_1^{n+1} = j_1^n + e^{dt}(-1+e^{ct})k_3^n(t) \\[2mm] j_2^{n+1} = (-1+e^{-bt})(-1+e^{ct})j_1^n + j_2^n + (-1+e^{-at})j_3^n + e^{dt}(-1+e^{-at})(-1+e^{-ct})k_1^n + \\ \qquad +e^{dt}(-1+e^{-at})(-1+e^{bt})k_2^n + e^{dt}(-1+e^{-bt})k_3^n \\[2mm] j_3^{n+1} = (-1+e^{at})(-1+e^{-bt})(-1+e^{ct})j_1^n + (-1+e^{at})j_2^n + j_3^n + e^{dt}(-1+e^{-ct})k_1^n + \\ \qquad +e^{dt}(-1+e^{bt})k_2^n + e^{dt}(-1+e^{at})(-1+e^{-bt})k_3^n \\[2mm] k_1^{n+1} = (-1+e^{ct})j_3^n + e^{dt}k_1^n \\[2mm] k_2^{n+1} = (-1+e^{-at})(-1+e^{ct})j_1^n + (-1+e^{-at})(-1+e^{-bt})j_2^n + (-1+e^{bt})j_3^n + \\ \qquad +e^{dt}(-1+e^{bt})(-1+e^{-ct})k_1^n + e^{dt}k_2^n + e^{dt}(-1+e^{-at})k_3^n \\[2mm] k_3^{n+1} = (-1+e^{ct})j_1^n + (-1+e^{-bt})j_2^n + (-1+e^{at})(-1+e^{bt})j_3^n + \\ \qquad +e^{dt}(-1+e^{at})(-1+e^{bt})(-1+e^{-ct})k_1^n + e^{dt}(-1+e^{at})k_2^n + e^{dt}k_3^n \\[2mm] p_1^{n+1} = e^{dt}(-1+e^{-bt})(-1+e^{-ct})k_2^n + p_1^n + (-1+e^{-bt})h^n \\[2mm] p_2^{n+1} = p_2^n + (-1+e^{-at})p_3^n + (-1+e^{-ct})h^n \\[2mm] p_3^{n+1} = (-1+E^(at))p_2^n + p_3^n + (-1+e^{at})(-1+e^{-ct})h^n \\[2mm] h^{n+1} = e^{dt}(-1+e^{-ct})k_2^n + (-1+e^{-bt})p_1^n + h^n \end{cases} \qquad (7)$$

Using MATHEMATICA the following can be proven:

**Theorem 4.1** *The Lie-Trotter integrator (7) has the following properties:*

*(i) It preserves the Poisson structure* $\Pi_-$*.*

*(ii) It preserves the Casimirs* $C_1, C_2$ *of our Poisson configuration* $(iso(3,1), \Pi_-)$*.*

*(iii) It does not preserve the Hamiltonian* $H$ *of the system (2).*

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