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SOME INEQUALITIES FOR ISOTONIC LINEAR FUNCTIONALS

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Abstract

In this paper is given a new variant of Minkowski-type inequality for isotonic linear functionals and then some variants of Qi's inequality for isotonic linear functionals using a new Young-type inequality. Also several applications are presented .[1]

Keywords and phrases: *Young's inequality, Minkowski's inequality, Qi's inequality.*

1 Introduction

In [1] are given new results which extend many generalizations of Young's inequality given before. We recall these results below in order to use them in the next sections.

Theorem 1.1 *Let λ, ν and τ be real numbers with $\lambda \geq 1$ and $0 < \nu < \tau < 1$. Then*

$$\left(\frac{\nu}{\tau}\right)^{\lambda} < \frac{A_{\nu}(a,b)^{\lambda} - G_{\nu}(a,b)^{\lambda}}{A_{\tau}(a,b)^{\lambda} - G_{\tau}(a,b)^{\lambda}} < \left(\frac{1-\nu}{1-\tau}\right)^{\lambda},$$

for all positive and distinct real numbers a and b. Moreover, both bounds are sharp.

The following important definition is given in [3], [5] and we will recall it here.

Let E be a nonempty set and L be a class of real-valued functions $f : E \to \mathbf{R}$ having the following properties:

(L1) If $f, g \in L$ and $a, b \in \mathbf{R}$, then $(af + bg) \in L$.

(L2) If $f(t) = 1$ for all $t \in E$, then $f \in L$.

[1] Mathematical Subject Classification(2008): *26D15, 26D10*

An *isotonic linear functional* is a functional $A : L \to \mathbf{R}$ having the following properties:

(A1) If $f, g \in L$ and $a, b \in \mathbf{R}$, then $A(af + bg) = aA(f) + bA(g)$.

(A2) If $f \in L$ and $f(t) \geq 0$ for all $t \in E$, then $A(f) \geq 0$.

The mapping A is said to be *normalised* if

(A3) $A(\mathbf{1}) = 1$.

The following Holder-type inequalities are obtained from Theorem 1.1 which is given in [1] and will be used in the next sections as an important tool in our demonstrations.

Theorem 1.2 *If L satisfy conditions L1, L2 and A satisfy A1, A2 on the set E. If $f^p,\ g^q,\ fg,\ f^{p\tau},\ g^{q(1-\tau)} \in L,\ A(f^p) > 0,\ A(g^q) > 0,\ p\tau > 1,\ \tau < 1,\ \frac{1}{p} + \frac{1}{q} = 1$ and f and g are positive functions then:*

$$\frac{1}{p\tau}\left[1 - \frac{A(f^{p\tau})A(g^{q(1-\tau)})}{A^{\tau}(f^p)A^{1-\tau}(g^q)}\right] < 1 - \frac{A(fg)}{A^{\frac{1}{p}}(f^p)A^{\frac{1}{q}}(g^q)} <$$

$$< \frac{1}{q(1-\tau)}\left[1 - \frac{A(f^{p\tau})A(g^{q(1-\tau)})}{A^{\tau}(f^p)A^{1-\tau}(g^q)}\right].$$

2 A refinement of Minkowski's inequality for isotonic linear functional

Using inequalities from Theorem 1.2 we can obtain some extensions of the classical Minkowski's inequality for isotonic linear functionals.

Theorem 2.1 *Let $1 < p_1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$, L satisfying conditions L1, L2 and A satisfying A1, A2 on the set E. Considering the nonnegative functions $f,\ h$ with $f^p,\ h^p,\ (f+h)^{\frac{p}{q_1}} f^{\frac{p}{p_1}},\ (f+h)^{\frac{p}{q_1}} h^{\frac{p}{p_1}},\ (f+h)^{p-1}f,\ (f+h)^{p-1}h \in L$ and $A(f^p) > 0,\ A(h^p) > 0,\ A((f+h)^p) > 0$ we will have,*

$$A^{\frac{1}{p}}((f+h)^p) < A^{\frac{1}{p}}(f^p)\left[1 - \frac{p_1}{p}\left(1 - \frac{A\left((f+h)^{\frac{p}{q_1}} f^{\frac{p}{p_1}}\right)}{A^{\frac{1}{q_1}}((f+h)^p)A^{\frac{1}{p_1}}(f^p)}\right)\right] +$$

$$+A^{\frac{1}{p}}(h^p)\left[1 - \frac{p_1}{p}\left(1 - \frac{A\left((f+h)^{\frac{p}{q_1}} h^{\frac{p}{p_1}}\right)}{A^{\frac{1}{q_1}}((f+h)^p)A^{\frac{1}{p_1}}(h^p)}\right)\right], \tag{1}$$

and

$$A^{\frac{1}{p}}(f^p)\left[1-\frac{q_1}{q}\left(1-\frac{A\left((f+h)^{\frac{p}{q_1}}f^{\frac{p}{p_1}}\right)}{A^{\frac{1}{q_1}}((f+h)^p)A^{\frac{1}{p_1}}(f^p)}\right)\right]+$$

$$+A^{\frac{1}{p}}(h^p)\left[1-\frac{q_1}{q}\left(1-\frac{A\left((f+h)^{\frac{p}{q_1}}h^{\frac{p}{p_1}}\right)}{A^{\frac{1}{q_1}}((f+h)^p)A^{\frac{1}{p_1}}(h^p)}\right)\right]<A^{\frac{1}{p}}((f+h)^p). \qquad (2)$$

Proof. We will check only inequality (1). Applying inequality from Theorem 1.2 first time for $f$and $\frac{(f+h)^{p-1}}{A^{\frac{1}{q}}((f+h)^{q(p-1)})}$ and then for h and $\frac{(f+h)^{p-1}}{A^{\frac{1}{q}}((f+h)^{q(p-1)})}$ we will have:

$$A^{\frac{1}{p}}((f+h)^p)=A\left(\frac{(f+h)^{p-1}}{A^{\frac{1}{q}}((f+h)^{q(p-1)})}(f+h)\right)=$$

$$=A\left(\frac{(f+h)^{p-1}}{A^{\frac{1}{q}}((f+h)^{q(p-1)})}f\right)+A\left(\frac{(f+h)^{p-1}}{A^{\frac{1}{q}}((f+h)^{q(p-1)})}h\right)\leq$$

$$<A^{\frac{1}{p}}(f^p)A^{\frac{1}{q}}\left(\frac{(f+h)^{q(p-1)}}{A((f+h)^{q(p-1)})}\right)\left[1-\frac{p_1}{p}\left(1-\frac{A\left((f+h)^{\frac{p}{q_1}}f^{\frac{p}{p_1}}\right)}{A^{\frac{1}{q_1}}((f+h)^p)A^{\frac{1}{p_1}}(f^p)}\right)\right]+$$

$$+A^{\frac{1}{p}}(h^p)A^{\frac{1}{q}}\left(\frac{(f+h)^{q(p-1)}}{A((f+h)^{q(p-1)})}\right)\left[1-\frac{p_1}{p}\left(1-\frac{A\left((f+h)^{\frac{p}{q_1}}h^{\frac{p}{p_1}}\right)}{A^{\frac{1}{q_1}}((f+h)^p)A^{\frac{1}{p_1}}(h^p)}\right)\right]=$$

$$=A^{\frac{1}{p}}(f^p)\left[1-\frac{p_1}{p}\left(1-\frac{A\left((f+h)^{\frac{p}{q_1}}f^{\frac{p}{p_1}}\right)}{A^{\frac{1}{q_1}}((f+h)^p)A^{\frac{1}{p_1}}(f^p)}\right)\right]+$$

$$+A^{\frac{1}{p}}(h^p)\left[1-\frac{p_1}{p}\left(1-\frac{A\left((f+h)^{\frac{p}{q_1}}h^{\frac{p}{p_1}}\right)}{A^{\frac{1}{q_1}}((f+h)^p)A^{\frac{1}{p_1}}(h^p)}\right)\right].$$

This result allow us to give a refinement of Minkowski's inequality for the cases of the time scales Cauchy delta, Cauchy nabla and α-diamond integrals.

Corollary 2.2 *(i) Let* $1<p_1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p_1}+\frac{1}{q_1}=1$, *and the positive functions* $f,h\in C_{rd}([a,b),\mathbf{R})$. *The following inequality takes place:*

$$\left(\int_a^b(f(x)+h(x))^p\Delta x\right)^{\frac{1}{p}}<$$

$$< \left(\int_a^b f^p(x)\Delta x\right)^{\frac{1}{p}} \left[1 - \frac{p_1}{p}\left(1 - \frac{\int_a^b (f(x)+h(x))^{\frac{p}{q_1}} f^{\frac{p}{p_1}}(x)\Delta x}{\left(\int_a^b (f(x)+h(x))^p \Delta x\right)^{\frac{1}{q_1}} \left(\int_a^b f^p(x)\Delta x\right)^{\frac{1}{p_1}}}\right)\right] +$$

$$+ \left(\int_a^b h^p(x)\Delta x\right)^{\frac{1}{p}} \left[1 - \frac{p_1}{p}\left(1 - \frac{\int_a^b (f(x)+h(x))^{\frac{p}{q_1}} h^{\frac{p}{p_1}}(x)\Delta x}{\left(\int_a^b (f(x)+h(x))^p \Delta x\right)^{\frac{1}{q_1}} \left(\int_a^b h^p(x)\Delta x\right)^{\frac{1}{p_1}}}\right)\right].$$

(ii) Let $1 < p_1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$, *and the positive functions* $f, h \in C_{ld}((a,b], \mathbf{R})$. *The following inequality takes place:*

$$\left(\int_a^b (f(x)+h(x))^p \nabla x\right)^{\frac{1}{p}} <$$

$$< \left(\int_a^b f^p(x)\nabla x\right)^{\frac{1}{p}} \left[1 - \frac{p_1}{p}\left(1 - \frac{\int_a^b (f(x)+h(x))^{\frac{p}{q_1}} f^{\frac{p}{p_1}}(x)\nabla x}{\left(\int_a^b (f(x)+h(x))^p \nabla x\right)^{\frac{1}{q_1}} \left(\int_a^b f^p(x)\nabla x\right)^{\frac{1}{p_1}}}\right)\right] +$$

$$+ \left(\int_a^b h^p(x)\nabla x\right)^{\frac{1}{p}} \left[1 - \frac{p_1}{p}\left(1 - \frac{\int_a^b (f(x)+h(x))^{\frac{p}{q_1}} h^{\frac{p}{p_1}}(x)\nabla x}{\left(\int_a^b (f(x)+h(x))^p \nabla x\right)^{\frac{1}{q_1}} \left(\int_a^b h^p(x)\nabla x\right)^{\frac{1}{p_1}}}\right)\right].$$

(iii) Let $1 < p_1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$ *and the positive functions* $f, h : [a,b] \to \mathbf{R}$ *be* $\diamond_\alpha$*-integrable functions. The following inequality takes place:*

$$\left(\int_a^b (f(x)+h(x))^p \diamond_\alpha x\right)^{\frac{1}{p}} <$$

$$< \left(\int_a^b f^p(x) \diamond_\alpha x\right)^{\frac{1}{p}} \left[1 - \frac{p_1}{p}\left(1 - \frac{\int_a^b (f(x)+h(x))^{\frac{p}{q_1}} f^{\frac{p}{p_1}}(x) \diamond_\alpha x}{\left(\int_a^b (f(x)+h(x))^p \diamond_\alpha x\right)^{\frac{1}{q_1}} \left(\int_a^b f^p(x) \diamond_\alpha x\right)^{\frac{1}{p_1}}}\right)\right] +$$

$$+ \left(\int_a^b h^p(x) \diamond_\alpha x\right)^{\frac{1}{p}} \left[1 - \frac{p_1}{p}\left(1 - \frac{\int_a^b (f(x)+h(x))^{\frac{p}{q_1}} h^{\frac{p}{p_1}}(x) \diamond_\alpha x}{\left(\int_a^b (f(x)+h(x))^p \diamond_\alpha x\right)^{\frac{1}{q_1}} \left(\int_a^b h^p(x) \diamond_\alpha x\right)^{\frac{1}{p_1}}}\right)\right].$$

3 Some variants of Qi's inequality for isotonic linear functionals

In this section we give several variants of some inequalities from [9] in the case of isotonic linear functionals for $p > 1$ using the corresponding Holder's inequalities from Theorem 1.2.

Lemma 3.1 *Let E, L and A be such that L1, L2, A1, A2 are satisfied.*

If $f,\ g,\ \frac{f^p}{g^{p-1}},\ f^{\frac{p}{p_1}} g^{1-\frac{p}{p_1}} \in L$ are positive functions with $A(g) > 0,\ A\left(\frac{f^p}{g^{p-1}}\right) > 0$ then

$$\left[1 - \frac{p_1}{p}\left(1 - \frac{A\left(f^{\frac{p}{p_1}} g^{1-\frac{p}{p_1}}\right)}{A^{\frac{1}{p_1}}\left(\frac{f^p}{g^{p-1}}\right) A^{\frac{1}{q_1}}(g)}\right)\right]^p A\left(\frac{f^p}{g^{p-1}}\right) > \frac{A^p(f)}{A^{p-1}(g)},$$

where $p > p_1 > 1$ with $\frac{1}{p_1} + \frac{1}{q_1} = 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We apply Holder' s inequality from Theorem 1.2 when $p > 1$ and $f,\ g,\ \frac{f^p}{g^{p-1}},\ f^{\frac{p}{p_1}} g^{1-\frac{p}{p_1}} \in L$ are positive functions, obtaining:

$$A(f) = A\left(\frac{f}{g^{\frac{1}{q}}} g^{\frac{1}{q}}\right) < A^{\frac{1}{p}}\left(\frac{f^p}{g^{\frac{p}{q}}}\right) A^{\frac{1}{q}}(g)\left[1 - \frac{p_1}{p}\left(1 - \frac{A\left(\frac{f^{\frac{p}{p_1}}}{g^{\frac{p}{p_1 q}}} g^{\frac{1}{q_1}}\right)}{A^{\frac{1}{p_1}}\left(\frac{f^p}{g^{\frac{p}{q}}}\right) A^{\frac{1}{q_1}}(g)}\right)\right].$$

Then we take the p-th power on both sides of the inequalities and have:

$$A^p(f) < A\left(\frac{f^p}{g^{\frac{p}{q}}}\right) A^{\frac{p}{q}}(g)\left[1 - \frac{p_1}{p}\left(1 - \frac{A\left(\frac{f^{\frac{p}{p_1}}}{g^{\frac{p}{p_1 q}}} g^{\frac{1}{q_1}}\right)}{A^{\frac{1}{p_1}}\left(\frac{f^p}{g^{\frac{p}{q}}}\right) A^{\frac{1}{q_1}}(g)}\right)\right]^p.$$

Theorem 3.2 *Let E, L and A be such that L1, L2, A1, A2 are satisfied. If $f,\ f^{p+2},\ f^{\frac{p+2}{p_1}} \in L$, f is positive function and $A(f) \geq A^2(\mathbf{1})$ then*

$$A^{p-1}(\mathbf{1}) A(f^{p+2})\left[1 - \frac{p_1}{p+2}\left(1 - \frac{A(f^{\frac{p+2}{p_1}})}{A^{\frac{1}{p_1}}(f^{p+2}) A^{\frac{1}{q_1}}(\mathbf{1})}\right)\right]^{p+2} > A^{p+1}(f),$$

takes place for $p + 2 > p_1 > 1$.

Proof. By Lemma 3.1 and hypothesis we have,

$$A(f^{p+2})\left[1-\frac{p_1}{p+2}\left(1-\frac{A(f^{\frac{p+2}{p_1}})}{A^{\frac{1}{p_1}}(f^{p+2})A^{\frac{1}{q_1}}(\mathbf{1})}\right)\right]^{p+2}=$$

$$=A\left(\frac{f^{p+2}}{1^{p+1}}\right)\left[1-\frac{p_1}{p+2}\left(1-\frac{A(f^{\frac{p+2}{p_1}})}{A^{\frac{1}{p_1}}(f^{p+2})A^{\frac{1}{q_1}}(\mathbf{1})}\right)\right]^{p+2}>$$

$$>\frac{A^{p+2}(f)}{A^{p+1}(\mathbf{1})}=\frac{A^{p+1}(f)A(f)}{A^{p-1}(\mathbf{1})A^{2}(\mathbf{1})}\geq\frac{A^{p+1}(f).}{A^{p-1}(\mathbf{1})}.$$

Consequence 3.3 *Let E, L and A be such that L1, L2, A1, A2 are satisfied. If f, f^{p+2}, $f^{\frac{p+2}{p_1}}\in L$, f is positive and in addition A is normalised and $A(f)\geq 1$ then*

$$A(f^{p+2})\left[1-\frac{p_1}{p+2}\left(1-\frac{A(f^{\frac{p+2}{p_1}})}{A^{\frac{1}{p_1}}(f^{p+2})}\right)\right]^{p+2}>A^{p+1}(f),$$

takes place for $p+2>p_1>1$.

As applications, we will give some refinements of several inequalities given by Qi and Yin, [9], in the cases of delta time-scale integral, the Cauchy nabla time-scales integrals and the Cauchy α-diamond time scale integrals.

Remark 3.4 *Let $a,b\in\mathbf{R}$, $a<b$. If $f\in C([a,b])$ is strictly positive and*

$$\int_a^b f(x)dx\geq(b-a)^2$$

then

$$\int_a^b f^{p+2}(x)dx\left[1-\frac{p_1}{p+2}\left(1-\frac{\int_a^b f^{\frac{p+2}{p_1}}(x)dx}{(b-a)^{\frac{1}{q_1}}(\int_a^b f^{p+2}(x)dx)^{\frac{1}{p_1}}}\right)\right]^{p+2}>$$

$$>\frac{1}{(b-a)^{p-1}}\left[\int_a^b f(x)dx\right]^{p+1},$$

where $p+2>p_1>1$.

Moreover, when we have delta time-scale integral, the Cauchy nabla time-scales integrals and the Cauchy α-diamond time scale integrals similary inequalities can be stated like above.

Lemma 3.5 *Let E, L and A be such that L1, L2, A1, A2 are satisfied on the set E. If $1 < p_1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$ and $f, g, f^{\frac{p}{p_1}} g^{1-\frac{p}{p_1}}, \frac{f^p}{g^{p-1}} \in L$ are positive functions and $A(\frac{f^p}{g^{p-1}}) > 0$, $A(g) > 0$ then*

$$\frac{A^p(f)}{A^{p-1}(g)} > A\left(\frac{f^p}{g^{p-1}}\right)\left[1 - \frac{q_1}{q}\left(1 - \frac{A(f^{\frac{p}{p_1}} g^{1-\frac{p}{p_1}})}{A^{\frac{1}{p_1}}\left(\frac{f^p}{g^{p-1}}\right) A^{\frac{1}{q_1}}(g)}\right)\right]^p.$$

Proof. We apply Holder's inequality from Theorem 1.2 when $g, f^{\frac{p}{p_1}} g^{1-\frac{p}{p_1}}, \frac{f^p}{g^{p-1}} \in L$ are positive functions, obtaining:

$$A(f) = A\left(\frac{f}{g^{\frac{1}{q}}} g^{\frac{1}{q}}\right) > A^{\frac{1}{p}}\left(\frac{f^p}{g^{\frac{p}{q}}}\right) A^{\frac{1}{q}}(g)\left[1 - \frac{q_1}{q}\left(1 - \frac{A\left(\left(\frac{f}{g^{\frac{1}{q}}}\right)^{\frac{p}{p_1}} g^{\frac{1}{q_1}}\right)}{A^{\frac{1}{p_1}}\left(\frac{f^p}{g^{\frac{p}{q}}}\right) A^{\frac{1}{q_1}}(g)}\right)\right].$$

Then we take the p-th power on both sides of the inequalities and we obtain by calculus the desired inequality.

The inequality from Lemma 3.5 can be written again for particular isotonic linear functionals, see for example [3] like below:

Consequence 3.6 *Let $a, b \in \mathbf{T}$, and $1 < p_1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$. If $f, g \in C_{rd}(\mathbf{T}, \mathbf{R})$ are positive then*

$$\frac{(\int_a^b f(x)\Delta x)^p}{(\int_a^b g(x)\Delta x)^{p-1}} > \int_a^b \frac{f^p(x)}{g^{p-1}(x)}\Delta x \left[1 - \frac{q_1}{q}\left(1 - \frac{\int_a^b f^{\frac{p}{p_1}}(x) g^{1-\frac{p}{p_1}}(x)\Delta x}{\left(\int_a^b \frac{f^p(x)}{g^{p-1}(x)}\Delta x\right)^{\frac{1}{p_1}} \left(\int_a^b g(x)\Delta x\right)^{\frac{1}{q_1}}}\right)\right]^p.$$

A new inequality for isotonic linear functional is the following:

Theorem 3.7 *Let E, L and A be such that L1, L2, A1, A2 are satisfied. If $f, f^p, f^{\frac{p_1}{p}} \in L$, f is positive, $A(f) > 0$ and $A(f) \geq A^{p-1}(\mathbf{1})$ then*

$$A^{p-1}(f) < A(f^p)\left[1 - \frac{p_1}{p}\left(1 - \frac{A\left(f^{\frac{p}{p_1}}\right)}{A^{\frac{1}{p_1}}(f^p) A^{\frac{1}{q_1}}(\mathbf{1})}\right)\right]^p$$

when $1 < p_1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$.

Proof. By Lemma 3.1 and hypothesis we have,

$$A(f^p) = A\left(\frac{f^p}{\mathbf{1}^{p-1}}\right) = A\left(\frac{f^p}{1^{\frac{p}{q}}}\right)$$

and

$$\frac{A^p(f)}{A^{p-1}(\mathbf{1})} < A\left(\frac{f^p}{\mathbf{1}^{p-1}}\right)\left[1-\frac{p_1}{p}\left(1-\frac{A\left(f^{\frac{p}{p_1}}\right)}{A^{\frac{1}{p_1}}(f)A^{\frac{1}{q_1}}(\mathbf{1})}\right)\right]^p$$

or

$$A^{p-1}(f) \leq \frac{A(f)A^{p-1}(f)}{A^{p-1}(\mathbf{1})} < A\left(f^p\right)\left[1-\frac{p_1}{p}\left(1-\frac{A\left(f^{\frac{p}{p_1}}\right)}{A^{\frac{1}{p_1}}(f)A^{\frac{1}{q_1}}(\mathbf{1})}\right)\right]^p.$$

If, in addition, the functional is normalised then previous inequality becomes:

Consequence 3.8 *Let E, L and A be such that L1, L2, A1, A2 are satisfied. If f, f^p, $f^{\frac{p_1}{p}} \in L$, f is positive, $A(f) > 0$, $A(f) \geq 1$ and in addition, A is normalised, then*

$$A^{p-1}(f) < A(f^p)\left[1-\frac{p_1}{p}\left(1-\frac{A\left(f^{\frac{p}{p_1}}\right)}{A^{\frac{1}{p_1}}(f^p)}\right)\right]^p$$

when $1 < p_1 < p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p_1}+\frac{1}{q_1}=1$.

As an application of Consequence 3.8 for Riemann integrals we obtain:

Consequence 3.9 *(i) Let $a, b \in \mathbf{R}$, $a < b$. If $f \in C([a,b])$ is positive, and*

$$\int_a^b f(x)dx \geq (b-a)^{p-1}$$

then

$$\left(\int_a^b f(x)dx\right)^{p-1} < \int_a^b f^p(x)dx\left[1-\frac{p_1}{p}\left(\frac{\int_a^b f^{\frac{p}{p_1}}(x)dx}{(b-a)^{\frac{1}{q_1}}(\int_a^b f(x)dx)^{\frac{1}{p_1}}}\right)\right]^p,$$

when $1 < p_1 < p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p_1}+\frac{1}{q_1}=1$.

(ii) In the case of delta time scale integrals, Cauchy nabla time-scales integrals and Cauchy α-diamond time scale integrals similary inequalities can be stated as above.

Now we take into account a particular case when A is a normalised functional and f, g are two applications such that $f, g : E \to \mathbf{R}$ like below, and we will obtain new variant of inequaliities from Lemma 3.5 and from [9] by using Theorem 2, see [7].

Lemma 3.10 *Let $A : L \to \mathbf{R}$ be an normalised isotonic linear functional and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f, g : E \to \mathbf{R}$ are so that $f, g, \frac{f^p}{g^{p-1}} \in L$ and $0 < m_1 \leq f \leq M_1 < \infty$, $0 < m_2 \leq g \leq M_2 < \infty$ for some constants m_1, M_1, m_2, M_2 then we have:*

$$\frac{A^p(f)}{A^{p-1}(g)} K^{Up} \left(\left(\frac{M_1}{m_1} \right)^p \left(\frac{M_2}{m_2} \right)^p \right) \geq A \left(\frac{f^p}{g^{p-1}} \right),$$

where $U = \max\{\frac{1}{p}, \frac{1}{q}\}$ and K is the Kantorovich's ratio defined by $K(h) = \frac{(h+1)^2}{4h}$, $h > 0$.

Proof. We use Theorem 2 from [7] where we replace f by $\frac{f}{g^{\frac{1}{q}}}$ and g by $g^{\frac{1}{q}}$ obtaining:

$$A(f) K^U \left(\left(\frac{M_1}{m_1} \right)^p \left(\frac{M_2}{m_2} \right)^p \right) = A \left(\frac{f^p}{g^{\frac{1}{q}}} g^{\frac{1}{q}} \right) K^U \left(\left(\frac{M_1}{m_1} \right)^p \left(\frac{M_2}{m_2} \right)^p \right) \geq$$

$$\geq A^{\frac{1}{p}} \left(\frac{f^p}{g^{p-1}} \right) A^{\frac{1}{q}}(g).$$

Now we take the p-th power on both sides of the inequalities and we get the conclusion.

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