SCIENTIFIC BULLETIN
OF
THE POLITEHNICA UNIVERSITY OF TIMIŞOARA

Vol 61 (75), Issue 2 MATHEMATICS –PHYSICS 2016

SOME INEQUALITIES FOR POWER SERIES WITH POSITIVE COEFFICIENTS

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Abstract

In this paper we use a technique given by Ibrahim, Dragomir and Mortici, in order to prove and enunciate several inequalities starting from some classical inequalities.

We present an improvement of Nesbitt's inequality and also a reverse of Nesbitt's inequality. Other important results which appear in the paper are some generalizations of well-known inequalities obtained by convergent power series with positive coefficients. [1]

1 Introduction

In [5], Ibrahim and Dragomir found some inequalities for power series via Buzano's result and some applications for several fundamental complex functions.

Ibrahim, Dragomir and Darus established in [6] some inequalities for power series with real coefficients by utilizing Young's inequality for sequences of complex numbers.

In [10], Mortici used the technique, by power series, for proving the well-known Nesbitt's inequality $\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{3}{2}$, $a,\ b,\ c>0$, which is equivalent to inequality, $\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}\geq\frac{3}{2}$, $a,\ b,\ c>0$, where $a+b+c=1$.

In demonstration, he used Jensen's inequality for the convex function $g(x)=x^n$ and geometric series, $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$, $|x|<1$. It is easy to see that $\frac{x}{1-x}=\sum_{n=1}^{\infty}x^n$, $|x|<1$.

In Theorem 1 are presented two inequalities which are used in Corollary 2 for an improvement of Nesbitt's inequality. Also another reverse of Nesbit's inequality is given in Corollary 3.

[1]Mathematical Subject Classification (2008): *26D15*
Keywords and phrases: *Power series, Young's inequality*

Other results which appear in this paper are some generalizations for convergent power series with positive coefficients of complements of Cauchy's inequality given in [9] starting from technique introduced in [10] and [6]. These results were obtained in Theorems 2, 3 and Theorem 4. By a similary technique we can also find a variant of reverse inequality of Young for functions which are sums of power series with positive coefficients in Proposition 2, using a result from [3].

2 Main results

Theorem 2.1 *For any $a \geq b \geq c > 0$ and $a+b+c=1$, there is the inequality*

$$\frac{1}{(1-c)^3}(a^2+b^2+c^2-\frac{1}{3}) \leq \frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}-\frac{3}{2} \leq$$
$$\leq \frac{1}{(1-a)^3}(a^2+b^2+c^2-\frac{1}{3}). \tag{1}$$

Proof. According to [4], if $g : I \to \mathbf{R}$ is a twice differentiable function such that there exist real constants γ and Γ so that $0 \leq \gamma \leq g''(x) \leq \Gamma$ for any $\in I$ we find the inequality

$$\frac{\gamma}{2}\sum_{j=1}^{3} p_j \left(x_j - \sum_{i=1}^{3} p_i x_i\right)^2 \leq \sum_{i=1}^{3} p_i g(x_i) - g\left(\sum_{i=1}^{3} p_i x_i\right) \leq \frac{\Gamma}{2}\sum_{j=1}^{3} p_j \left(x_j - \sum_{i=1}^{3} p_i x_i\right)^2,$$

where $p_i > 0$ for all $i \in \{1,2,3\}$ and $\sum_{i=1}^{3} p_i = 1$.

Since $a \geq b \geq c > 0$ and the function $g(x) = x^n,\ n \geq 2$, is convex and $p_1 = p_2 = p_3 = \frac{1}{3}$, implies

$$0 \leq \gamma = g''(c) = n(n-1)c^{n-2} \leq g''(x) \leq \Gamma = g''(a) = n(n-1)a^{n-2}.$$

Therefore we have the following inequality

$$\frac{n(n-1)c^{n-2}}{6}\sum_{cyclic}\left(a - \frac{a+b+c}{3}\right)^2 \leq \frac{a^n+b^n+c^n}{3} - \left(\frac{a+b+c}{3}\right)^n \leq$$
$$\leq \frac{n(n-1)a^{n-2}}{6}\sum_{cyclic}\left(a - \frac{a+b+c}{3}\right)^2.$$

Because $a,\ b,\ c > 0$ and $a+b+c=1$, we deduce the inequality

$$\frac{n(n-1)c^{n-2}}{2}\sum_{cyclic}\left(a-\frac{1}{3}\right)^2 \leq a^n+b^n+c^n-3\left(\frac{1}{3}\right)^n \leq \frac{n(n-1)a^{n-2}}{2}\sum_{cyclic}\left(a-\frac{1}{3}\right)^2.$$

By passing to power series we obtain

$$\frac{1}{2}\sum_{cyclic}\left(a-\frac{1}{3}\right)^2\sum_{n=1}^{\infty}n(n-1)c^{n-2}\le\sum_{n=1}^{\infty}a^n+\sum_{n=1}^{\infty}b^n+\sum_{n=1}^{\infty}c^n-3\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n\le$$

$$\le\frac{1}{2}\sum_{cyclic}\left(a-\frac{1}{3}\right)^2\sum_{n=1}^{\infty}n(n-1)a^{n-2}.$$

But, we know the power series $\frac{x}{1-x}=\sum_{n=1}^{\infty}x^n$, and $\frac{2}{(1-x)^3}=\sum_{n=1}^{\infty}n(n-1)x^n$, $|x|<1$. Therefore, the above inequality becomes

$$\frac{1}{(1-c)^3}\sum_{cyclic}\left(a-\frac{1}{3}\right)^2\le\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}-\frac{3}{2}\le\frac{1}{(1-a)^3}\sum_{cyclic}\left(a-\frac{1}{3}\right)^2,$$

which is equivalent to the inequality

$$\frac{1}{(1-c)^3}\left(a^2+b^2+c^2-\frac{1}{3}\right)\le\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}-\frac{3}{2}\le$$

$$\le\frac{1}{(1-a)^3}\left(a^2+b^2+c^2-\frac{1}{3}\right).$$

The below inequality represents an improvement of Nesbitt's inequality.

Corollary 2.2 *For any $a\ge b\ge c>0$, there is the inequality*

$$\frac{a+b+c}{3(a+b)^3}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}\le$$

$$\le\frac{a+b+c}{3(b+c)^3}\left[(a-b)^2+(b-c)^2+(c-a)^2\right].\qquad(2)$$

Proof. In Theorem 1 we assume, without loss of generality, that $a+b+c=1$. By replacement in inequality (2), we deduce inequality (1). Therefore, the requirement is true.

Another reverse inequality of Nesbitt's inequality is the following:

Corollary 2.3 *For any $a\ge b\ge c>0$, there is the inequality*

$$0\le\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}-\frac{3}{2}\le3\left(\frac{a}{b+c}+\frac{c}{a+b}-2\frac{a+c}{a+2b+c}\right).\qquad(3)$$

Proof. We assume, without loss of generality, that $a+b+c=1$. By replacement in inequality (3), we deduce inequality

$$0 \leq \frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}-\frac{3}{2} \leq 3\left(\frac{a}{1-a}+\frac{c}{1-c}-2\frac{a+c}{2-a-c}\right).$$

Simić showed in [11], that if $(x_i)_{i=1}^n \in [a,b]^n$, then

$$0 \leq \sum_{i=1}^n p_i g(x_i) - g\left(\sum_{i=1}^n p_i x_i\right) \leq g(a)+g(b)-2g\left(\frac{a+b}{2}\right),$$

where $p_i>0$ for all $i \in \{1,...,n\}$ to see that $\sum_{i=1}^n p_i = 1$.

Since $a \geq b \geq c > 0$ and the function $g(x)=x^n$ is convex, and $p_1=p_2=p_3=\frac{1}{3}$, it follows the inequality

$$0 \leq \frac{a^n+b^n+c^n}{3} - \left(\frac{a+b+c}{3}\right)^n \leq a^n+c^n-2\left(\frac{a+c}{2}\right)^n,$$

so, by passing to power series, we deduce

$$0 \leq \sum_{n=1}^{\infty} a^n + \sum_{n=1}^{\infty} b^n + \sum_{n=1}^{\infty} c^n - 3\sum_{n=1}^{\infty}\left(\frac{a+b+c}{3}\right)^n \leq 3\left[\sum_{n=1}^{\infty} a^n + \sum_{n=1}^{\infty} c^n - 2\sum_{n=1}^{\infty}\left(\frac{a+c}{2}\right)^n\right],$$

which is equivalent to

$$0 \leq \frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}-\frac{3}{2} \leq 3\left(\frac{a}{1-a}+\frac{c}{1-c}-2\frac{a+c}{2-a-c}\right),$$

where $a,\ b,\ c>0$ and $a+b+c=1$.

We consider as in [6], an analytic function defined by the power series

$$f(z)=\sum_{n=0}^{\infty} a_n z^n$$

with real coefficients and convergent on the unit disk $D(0,R),\ R>0$. Let $f_A(z)$ is a new power series defined by $\sum_{n=0}^{\infty} |a|_n z^n$ where $a_n=|a_n| sgn(a_n)$ where $sgn(x)$ is the real signum function as in [6]. The power series $f_A(z)$ has the same radius of convergence as the original power series $f(z)$.

Also, in [3], in Corollary 2.2, (ii) the authors presented as an alternative reverse inequality for Young's inequality two inequalities. Using one of these inequalities we can find the following inequality for functions which are sums of power series with positive coefficients.

Proposition 2.4 *Let $f(z)$ be an analytic function defined by the power series $\sum_{n=0}^{\infty} a_n z^n$ with real coefficients and convergent on the open disk $D(0,R) \subset \mathbf{C}$, and f_A as in [6].*

For $|a|, |b| \in (0,R)$ and $\lambda \in [0,1]$, the following inequality holds:

$$|(1-\lambda)f(|a|) + \lambda f(|b|)| \leq (1-\lambda)f_A(|a|) + \lambda f_A(|b|) \leq$$

$$\leq f_A(|a|^{1-\lambda}|b|^{\lambda}) + \lambda(1-\lambda)\left[\log\left(\frac{|a|}{|b|}\right)\right]^2 \cdot [d f_A'(d) + d^2 f_A''(d)],$$

where $d = \max\{|a|, |b|\}$.

Proof We can suppose without loss of generality that $d = \max\{a,b\} = a$. Then for every $n \in \mathbf{N}^*$ we see that $d^n = \max\{a^n, b^n\}$. By replacing a with $|a|^n$, b with $|b|^n$ and d with d^n in $a^{1-\lambda}b^{\lambda} \leq (1-\lambda)a + \lambda b \leq a^{1-\lambda}b^{\lambda} + \lambda(1-\lambda)\left[\log\left(\frac{a}{b}\right)\right]^2 d$ and then multiplying by $a_n \geq 0$ for every $n \in \mathbf{N}^*$ we get

$$|a_n(1-\lambda)a^n + a_n\lambda b^n| \leq |a_n|(1-\lambda)|a|^n + |a_n|\lambda|b|^n \leq$$

$$\leq |a_n||a|^{n(1-\lambda)}|b|^{n\lambda} + |a_n|\lambda(1-\lambda)\left[\log\left(\frac{|a|^n}{|b|^n}\right)\right]^2 |d|^n,$$

for every $n \in \mathbf{N}^*$. Then by adding previous inequalities when $n \in \{1,2,...,m\}$ and $m \in \mathbf{N}^*$ we obtain,

$$|\sum_{n=1}^{m} a_n(1-\lambda)a^n + \sum_{n=1}^{m} a_n\lambda b^n| \leq \sum_{n=1}^{m}(1-\lambda)|a_n||a|^n + \sum_{n=1}^{m}\lambda|a_n||b|^n \leq$$

$$\leq \sum_{n=1}^{m}|a_n|[|a|^{(1-\lambda)}|b|^{\lambda}]^n + \lambda(1-\lambda)\left[\log\left(\frac{|a|}{|b|}\right)\right]^2 \sum_{n=1}^{m}|a_n|n^2 d^n.$$

When m tends to infinity we have

$$(1-\lambda)f_A(|a|) + \lambda f_A(|b|) \leq f_A(|a|^{1-\lambda}|b|^{\lambda}) + \lambda(1-\lambda)\left[\log\left(\frac{|a|}{|b|}\right)\right]^2 S(d),$$

because $0 < |a| < R,\ 0 < |b| < R,\ 0 < |a|^{1-\lambda}|b|^{\lambda} < R$ and $0 < d < R$.

In this case $S(z)$ is the sum of the convergent series $\sum_{n=1}^{\infty} a_n n^2 z^n$ for $|z| \in D(0,R)$ and is $z f_A'(z) + z^2 f_A''(z)$. This series has the same convergence radius as series which has the sum $f(z)$.

Proposition 2.5 *Let $f : [0, \infty) \to \mathbf{R}$ is any increasing and concave function, and $x, y, z \in D(0, R) \subset \mathbf{C}^*$ with $0 < |x| \leq |y| \leq |z|$. If $g(z)$ is an analytic function defined by the power series $\sum_{n=1}^{\infty} a_n z^n$ with real coefficients and is convergent on the open disk $D(0, R) \subset \mathbf{C}$ and g_A is as in [6] then the following inequalities hold:*

$$g_A(|z|)f(|y|) + g_A(|y|)f(|x|) + g_A f(|x|)f(|z|) \geq$$

$$\geq g_A(|x|)f(|y|) + g_A(|z|)f(|x|) + g_A(|y|)f(|z|)$$

and

$$|z|g_A'(|z|)f(|y|) + |y|g_A'(y)f(|x|) + |x|g_A'(|x|)f(|z|) \geq$$

$$\geq |x|g_A'(|x|)f(|y|) + |z|g_A'(|z|)f|(|x|) + |y|g_A'(|y|)f(|z|),$$

if in addition, g_A is a differentiable mapping on $D(0, R)$.

Proof.
In [7], the author showed that

$$(z^n - x^n)f(y) \geq (z^n - y^n)f(x) + (y^n - x^n)f(z),$$

where $f : [0, \infty) \to \mathbf{R}$ is any increasing and concave function, $0 < x \leq y \leq z$, and n is a positive integer.

Multiplying by $a_n \geq 0$ last inequality for $n = 1, 2, ...$ with $0 < |x| \leq |y| \leq |z| < R$, and then summing with respect to n from 1 to m, when m tends to infinity, we get

$$(g(|z|) - g(|x|))f(|y|) \geq (g(|z|) - g(|y|))f(|x|) + (g(|y|) - g(|x|))f(|z|),$$

or

$$g(|z|)f(|y|) + g(|y|)f(|x|) + g(|x|)f(|z|) \geq g(|x|)f(|y|) + g(|z|)f(|x|) + g(|y|)f(|z|),$$

where $g(x) = \sum_{n=1}^{\infty} a_n x^n$ and $a_n \geq 0,\ (\forall)\ n \in \mathbf{N}^*$.

Now if we multiply the inequality given in [7] by $na_n \geq 0$ for $n = 1, 2, ...$ and then summing with respect to n, we have

$$(|z|g'(|z|) - |x|g'(|x|))f(|y|) \geq (|z|g'(|z|) - |y|g'(|y|))f(|x|) + (|y|g'(|y|) - |x|g'(|x|))f(|z|)$$

where $g(x) = \sum_{n=1}^{\infty} a_n x^n$ with $a_n \geq 0,\ (\forall)\ n \in \mathbf{N}^*$, g_A is a differentiable mapping on $D(0, R)$.

3 Generalizations of several well-known inequalities

In the following we will give a generalization of a complement of Cauchy's inequality given in [2] and [9] by J. B. Diaz and F. T. Metcalf for power series with positive coefficients.

Theorem 3.1 *Let $f(z)$ be an analytic function defined by the power series $\sum_{n=0}^{\infty} a_n z^n$ with real coefficients and convergent on the open disk $D(0,R) \subset \mathbf{C}$, and f_A as in [6].*

(i) If d, b are n-tuples which satisfy conditions $0 < m_1 \leq |d_i| \leq M_1$ and $0 < m_2 \leq |b_i| \leq M_2$, $d_i,\ b_i \in \mathbf{C}$, $(i = \overline{1,n})$ for some constants $m_1,\ m_2,\ M_1$ and M_2 then we have,

$$|\sum_{k=1}^{n} f(b_k^2) + \sum_{k=1}^{n} f\left(\frac{m_2}{M_1}\frac{M_2}{m_1} d_k^2\right)| \leq$$

$$\sum_{k=1}^{n} f_A(|b_k|^2) + \sum_{k=1}^{n} f_A\left(\frac{m_2}{M_1}\frac{M_2}{m_1}|d_k|^2\right) \leq \sum_{k=1}^{n} f_A\left(\frac{M_2}{m_1}|d_k b_k|\right) + \sum_{k=1}^{n} f_A\left(\frac{m_2}{M_1}|d_k b_k|\right).$$

when $M_2^2 \frac{M_1}{m_1} < R$.

(ii) Under previous conditions, if in addition f is a differentiable mapping on $D(0,R)$ we obtain,

$$|\sum_{k=1}^{n} b_k^2 f'(b_k^2) + \sum_{k=1}^{n} \frac{m_2}{M_1}\frac{M_2}{m_1} d_k^2 f'(\frac{m_2}{M_1}\frac{M_2}{m_1} d_k^2)| \leq$$

$$\leq \sum_{k=1}^{n} |b_k|^2 f_A'(|b_k|^2) + \sum_{k=1}^{n} \frac{m_2}{M_1}\frac{M_2}{m_1}|d_k|^2 f_A'\left(\frac{m_2}{M_1}\frac{M_2}{m_1}|d_k|^2\right) \leq$$

$$\leq \sum_{k=1}^{n} \frac{M_2}{m_1}|d_k b_k| f_A'\left(\frac{M_2}{m_1}|d_k b_k|\right) + \sum_{k=1}^{n} \frac{m_2}{M_1}|d_k b_k| f_A'\left(\frac{m_2}{M_1}|d_k b_k|\right).$$

Proof. (i) Taking into account inequality,

$$\sum_{k=1}^{n} b_k^2 + \frac{m_2}{M_1}\frac{M_2}{m_1}\sum_{k=1}^{n} d_k^2 \leq \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right)\sum_{k=1}^{n} d_k b_k,$$

see [9], inequality (7), where d_k is replaced by $|d_k|^p$, b_k is replaced by $|b_k|^p$, m_1 is replaced by m_1^p, M_1 by M_1^p, m_2 by m_2^p and M_2 by M_2^p, $p \in \mathbf{N}^*$ and multiplying by $|a_p|$ the inequality before summing we obtain

$$|\sum_{p=1}^{m} a_p \left(\sum_{k=1}^{n} b_k^{2p} + \frac{m_2^p}{M_1^p}\frac{M_2^p}{m_1^p}\sum_{k=1}^{n} d_k^{2p}\right)| \leq$$

$$\leq \sum_{p=1}^{m} |a_p| \left(\sum_{k=1}^{n} |b_k|^{2p} + \frac{m_2^p}{M_1^p} \frac{M_2^p}{m_1^p} \sum_{k=1}^{n} |d_k|^{2p} \right) \leq \sum_{p=1}^{m} |a_p| \left[\left(\frac{M_2^p}{m_1^p} + \frac{m_2^p}{M_1^p} \right) \sum_{k=1}^{n} |d_k b_k|^p \right].$$

Using hypothesis $0 < m_1 \leq |d_i| \leq M_1$ and $0 < m_2 \leq |b_i| \leq M_2$ ($i = \overline{1,n}$), when $M_2^2 \frac{M_1}{m_1} < R$ we notice that $|b_k|^2$, $\frac{m_2}{M_1} \frac{M_2}{m_1} |d_k|^2$, $\frac{M_2}{m_1} |d_k b_k|$ and $\frac{m_2}{M_1} |d_k b_k|$ are in $D(0,R)$ and then the power series being convergent, we obtain the inequality from conclusion.

An improvement of last theorem, using Theorem 2 from [9], will be also presented below:

Theorem 3.2 *Let $f(z)$ be an analytic function defined by the power series $\sum_{n=0}^{\infty} a_n z^n$ with real coefficients and convergent on the open disk $D(0,R) \subset \mathbf{C}$, and f_A as in [6]. Let u, v be real numbers such that $0 < v \leq u < 1$, $u + v = 1$ and w positive n-tuple and d, b n-tuples of complex numbers such that $0 \leq m \leq \frac{|b_k|}{|d_k|} \leq M$, and $|d_k|^2 M^2 < R$ $(k = 1, ..., n)$. Then*

$$|u \sum_{k=1}^{n} w_k f(b_k^2) + v \sum_{k=1}^{n} w_k f(M m d_k^2)| \leq u \sum_{k=1}^{n} w_k f_A(|b_k|^2) + v \sum_{k=1}^{n} w_k f_A(M m |d_k|^2) \leq$$

$$\leq v \sum_{k=1}^{n} w_k f_A(m |d_k b_k|) + u \sum_{k=1}^{n} w_k f_A(M |d_k b_k|).$$

Proof. Using the inequality,

$$u \sum_{k=1}^{n} w_k b_k^2 + v M m \sum_{k=1}^{n} w_k d_k^2 \leq (vm + uM) \sum_{k=1}^{n} w_k d_k b_k,$$

see [9], inequality (8), where d_k is replaced by $|d_k|^p$, b_k is replaced by $|b_k|^p$, m is replaced by m^p, M by M^p, $p \in \mathbf{N}^*$ and multiplying by a_p the inequality obtained before summing we obtain

$$\sum_{p=1}^{n_1} a_p \left(u \sum_{k=1}^{n} w_k |b_k|^{2p} + v M^p m^p \sum_{k=1}^{n} w_k |d_k|^{2p} \right) \leq \sum_{p=1}^{n_1} a_p \left[(v m^p + u M^p) \sum_{k=1}^{n} w_k |d_k b_k|^p \right].$$

By hypothesis, $m \leq \frac{|d_k|}{|b_k|} \leq M$, and $|d_k|^2 M^2 < R$, $(k = \overline{1,n})$ we notice that $|b_k|^2 < R$, $M m |d_k|^2 < R$, $m |d_k b_k| < R$ and $M |d_k b_k| < R$ $(k = \overline{1,n})$ and thus previous inequality becomes

$$u \sum_{k=1}^{n} w_k f_A(|b_k|^2) + v \sum_{k=1}^{n} w_k f_A(M m |d_k|^2) \leq v \sum_{k=1}^{n} w_k f_A(m |d_k b_k|) + u \sum_{k=1}^{n} w_k f_A(M |d_k b_k|),$$

when n_1 tends to infinity.

Theorem 3.3 *Let d and b be two n-tuples of complex numbers, $p^{-1}+q^{-1}=1,\ 0<m<M,\ 0\le m\le \frac{|d_i|}{|b_i|^{\frac{q}{p}}}\le M,\ \ (i=1,...,n),\ p_i\ge 0,\ (i=1,...,n)$. If $p>1$ then we have,*

$$|\sum_{k=1}^{n} p_k[f(Md_k^p)-f(md_k^p)]+\sum_{k=1}^{n} p_k[f(mM^p b_k^q)-f(Mm^p b_k^q)]|\le$$

$$\le \sum_{k=1}^{n} p_k[f_A(M|d_k|^p)-f_A(m|d_k|^p)]+\sum_{k=1}^{n} p_k[f_A(mM^p|b_k|^q)-f_A(Mm^p|b_k|^q)]\le$$

$$\le \sum_{k=1}^{n} p_k[f_A(M^p|d_k b_k|)-f_A(m^p|d_k b_k|)],$$

where $f(z)$ is the sum of the power series $\sum_{n=1}^{\infty} a_n z^n$ and f and f_A are as in [6].

Proof. This time we will use the inequality (9) from Theorem 5, see [9] where d_k will be replaced by $|d_k|^r$, b_k by $|b_k|^r$, m by m^r and M by M^r, $r\in\{1,2,...\}$ and the proof will be like before.

Remark 3.4 *For example, under the conditions of previous theorems, these inequalities can be stated for the functions like, $e^x,\ \sinh(x),\ \cosh(x),\ \frac{1}{2}\ln\frac{1+x}{1-x}, |x|<1$ or $\frac{1+x}{4x}+\frac{1-x^2}{2x^2}ln(1-x),\ x\in(-1,0)\cup(0,1)$.*

Remark 3.5 *As in [6], there exist some inequalities for special functions such as polylogarithm, hypergeometric, Bessel and modified Bessel functions for the first kind. It is known that $Li_n(z)$, ${}_2F_1(a,b;c;z)$, $J_a(z)$ and $I_a(z)$ are power series with real coefficients and convergent on the open disk $D(0,1)$. Therefore, like in [6], we can think to rewrite the inequalities given before under conditions from our theorems. For example, if $Li_n(z)$ is the polylogarithm function, then we have*

$$Li_n(a^{1-\lambda}b^{\lambda})\le(1-\lambda)Li_n(a)+\lambda Li_n(b)\le$$

$$\le Li_n(a^{1-\lambda}b^{\lambda})+\lambda(1-\lambda)\left[\log\frac{a}{b}\right]^2\left[dLi_n^{'}(d)+d^2Li_n^{''}(d)\right],$$

where $d=\max\{a,b\}$, $a,b\in D(0,1)$, $\lambda\in[0,1]$.

Remark 3.6 *Let R be the converence radius of the power series $\sum_{n=0}^{\infty} a_n x^n$ with positive coefficients, which has the sum $f(x)$ on $(-R, R)$.*

(a) If $a,\ b,\ c \in (-\sqrt{R}, \sqrt{R})$ then there is the inequality

$$f_A(a^2) + f_A(b^2) + f_A(c^2) \geq f_A(ab) + f_A(bc) + f_A(ca). \tag{4}$$

(b) If $x_1,\ x_2, ..., x_n \in (-\sqrt{R}, \sqrt{R})$ then there is the inequality

$$f_A(x_1^2) + f_A(x_2^2) + ... + f_A(x_n^2) \geq f_A(x_1 x_2) + f_A(x_2 x_3) + ... + f_A(x_{n-1} x_n) + f_A(x_n x_1). \tag{5}$$

Proof.

(a) If $a, b, c \in (-\sqrt{R}, \sqrt{R})$, then taking into account the well-known inequality, $a^2 + b^2 + c^2 \geq ab + bc + ca$, which is true for all $a, b, c \in \mathbf{R}$ for every $n \in \mathbf{N}$ we have,

$$a^{2n} + b^{2n} + c^{2n} \geq (ab)^n + (bc)^n + (ca)^n$$

when a will be replaced by a^n, b is replaced by b^n and c is replaced by c^n, and then

$$\sum_{n=1}^{m} a_n (a^{2n} + b^{2n} + c^{2n}) \geq \sum_{n=1}^{m} a_n ((ab)^n + (bc)^n + (ca)^n).$$

When m tends to infinity we get the inequality from (a).

4 Applications

1. We know the following power series: $\log\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} \frac{1}{n} x^n$, for any $|x| < 1$.

Therefore, if we take the function $f(x) = \log\left(\frac{1}{1-x}\right)$, then using inequality (4), we obtain the following inequality:

$$(1 - ab)(1 - bc)(1 - ca) \geq (1 - a^2)(1 - b^2)(1 - c^2), \tag{6}$$

for every $a, b, c \in (-1, 1)$.

This inequality is equivalent to the inequality

$$a^2 + b^2 + c^2 + abc(a + b + c) \geq ab + bc + ca + a^2 b^2 + b^2 c^2 + c^2 a^2, \tag{7}$$

for every $a, b, c \in (-1, 1)$, which implies the following inequality:

$$a^2 + b^2 + c^2 \geq ab + bc + ca + \frac{1}{2}[a^2(b - c)^2 + b^2(c - a)^2 + c^2(a - b)^2], \tag{8}$$

for every $a, b, c \in (-1, 1)$.

Inequality (8) proved the inequality

$$a^2 + b^2 + c^2 \geq ab + bc + ca + \frac{a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2}{2(a+b+c)^2}, \tag{9}$$

for every $a, b, c > 0$, which implies the following inequality

$$3(a^2 + b^2 + c^2) \geq (a+b+c)^2 + \frac{a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2}{(a+b+c)^2}, \tag{10}$$

for any $a, b, c > 0$.

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