SCIENTIFIC BULLETIN
OF
THE "POLITEHNICA" UNIVERSITY OF TIMIŞOARA

Vol 61 (75), Issue 1 MATHEMATICS –PHYSICS 2016

ON INTERPOLATION OF LOCALLY CONVEX COUPLES WITH REAL METHODS

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Abstract

We consider a general form of Peetre's $\mathcal{K}$ - and $\mathcal{J}$ - methods of interpolation for locally convex couples. [1]

Keywords and phrases: *interpolation methods, locally convex couple, interpolation operator, Haussdorff locally convex space.*

1 Introduction

In this paper we present a general form of Peetre's $\mathcal{K}$ - and $\mathcal{J}$ - methods of interpolation for locally convex couples. A theorem on interpolation of bilinear operators is given.

2 Preliminaries

Let X_0 and X_1 be two Haussdorff locally convex spaces, so that $X_0 \cap X_1 \neq \{\theta\}$. Then we shall say that $\vec{X} = (X_0, X_1)$ is a locally convex couple if there is a Haussdorff topological vector space $\mathcal{E}$ so that the space X_0 and X_1 are embleded linearly and continously in $\mathcal{E}$ $(X_i \hookrightarrow \mathcal{E}, i = 0, 1)$.

Everywhere in this paper we suppose that the topology of $X_i, i = 0, 1$ is generate by the families of semi-norms $\{p_{j\alpha^j}\}_{\alpha^j \in \mathcal{A}_j}$, $j = 0, 1$, which is directed and filled completed.

We will note by

$$p^{\Delta}_{\alpha^0,\alpha^1}(x) = max \{p_{0\alpha^0}(x), p_{1\alpha^1}(x)\}, \quad x \in X_0 \cap X_1$$

[1]MSC (2010): 46M35

$$p^{\Sigma}_{\alpha^0,\alpha^1}(x) = inf\left\{p_{0\alpha^0}(x_0) + p_{1\alpha^1}(x_1) : x = x_0 + x_1, x_i \in X_i, i = 0,1\right\}$$

Then

i) $\Delta(\overrightarrow{X}) = X_0 \cap X_1$ is a Haussdorff locally convex space, whose topology is generated by the family of semi-norms $\left\{p^{\Delta}_{\alpha^0,\alpha^1}\right\}_{(\alpha_0,\alpha_1)\in\mathcal{A}_0\times\mathcal{A}_1}$.

ii) $\Sigma(\overrightarrow{X}) = X_0 + X_1$ is a Haussdorff locally convex space, whose topology is generated by the family of semi-norms $\left\{p^{\Sigma}_{\alpha^0,\alpha^1}\right\}_{(\alpha_0,\alpha_1)\in\mathcal{A}_0\times\mathcal{A}_1}$.

The family of semi-norms $\left\{p^{\Delta}_{\alpha^0,\alpha^1}\right\}_{(\alpha_0,\alpha_1)\in\mathcal{A}_0\times\mathcal{A}_1}$ defines the least fine topology for which the canonical maps $i_k : X_0 \cap X_1 \to X_k, k = 0,1$ are continuous.

The family of semi-norms $\left\{p^{\Sigma}_{\alpha^0,\alpha^1}\right\}_{(\alpha_0,\alpha_1)\in\mathcal{A}_0\times\mathcal{A}_1}$ defines the fines topology for which the canonical maps $j_k : X_k \to X_0 + X_1, k = 0,1$ are continuous.

Thus we can write

$$X_0 \cap X_1 \hookrightarrow X_k \hookrightarrow X_0 + X_1, \quad k = 0,1.$$

We denote

$$S(p_{i\alpha_i}, r) = \{x \in X_i : p_{i\alpha_i}(x) < r\}, i = 0,1,$$
$$S(p^{\Delta}_{\alpha_0,\alpha_1}, r) = \{x \in X_0 \cap X_1 : p^{\Delta}_{\alpha_0,\alpha_1}(x) < r\},$$
$$S(p^{\Sigma}_{\alpha_0,\alpha_1}, r) = \{x \in X_0 + X_1 : p^{\Sigma}_{\alpha_0,\alpha_1}(x) < r\},$$

and by $|co|M$ absolutely convex hull of a set M.

Proposition 2.1

We have

(a) $S(p^{\Delta}_{\alpha_0,\alpha_1}, r) = S(p_{0\alpha_0}, r) \cap S(p_{1\alpha_1}, r)$

(b) $S(p^{\Sigma}_{\alpha_0,\alpha_1}, r) = |co|(S(p_{0\alpha_0}, r), S(p_{1\alpha_1}, r))$

In this paper $T \in \mathcal{L}(X,Y)$ means that T is a bounded linear mapping from X into Y and $T \in \mathcal{L}(\overrightarrow{X}, \overrightarrow{Y})$ means that T is a linear mapping from X_0+X_1 into Y_0+Y_1 so that $T|_{X_i} : X_i \to Y_i$ is bounded (here $T|_X$ denotes the restriction of T into X).

Proposition 2.2

Let $\overrightarrow{X} = (X_0, X_1)$, $\overrightarrow{Y} = (Y_0, Y_1)$ *be two locally convex couples and* $T \in \mathcal{L}(\overrightarrow{X}, \overrightarrow{Y})$.

(a) Then $T : \Sigma(\overrightarrow{X}) \longrightarrow \Sigma(\overrightarrow{Y})$ and $T : \Delta(\overrightarrow{X}) \longrightarrow \Delta(\overrightarrow{Y})$ *are bounded maps;*

(b) If in addition $T|_{X_i} : X_i \longrightarrow Y_i$, $i = 0, 1$ are completed bounded then $T : \Sigma(\vec{X}) \longrightarrow \Sigma(\vec{Y})$ is complete bounded;

(c) If in addition $T|_{X_i} : X_i \longrightarrow Y_i$, $i = 0, 1$ are compact then $T : \Sigma(\vec{X}) \longrightarrow \Sigma(\vec{Y})$ is compact.

3 The real interpolation method

Let us begin by remembering some basic notation. Let $\vec{X} = (X_0, X_1)$ be a locally convex couple and $t > 0$.

The Peetre $K-$ functional, $\mathcal{K}_{\alpha^0,\alpha^1}$, $\alpha^i \in \mathcal{A}_i$, $i = 0, 1$, is defined for $x \in \Sigma(\vec{X})$ by

$$\mathcal{K}_{\alpha^0,\alpha^1}(t, x, \vec{X}) = inf\{p_{0\alpha^0}(x_0) + tp_{1\alpha^1}(x_1), x = x_0 + x_1, x_i \in X_i\}$$

Similarity the $J-$ functional,

$$\mathcal{J}_{\alpha^0,\alpha^1}(t, x, \vec{X}) = max(p_{0\alpha^0}(x), p_{1\alpha^1}(x))$$

Let E be a $\mathbb{Z}$-lattice, i.e. E is quasi-Banach space of two-sided numerical sequences $a = (a_n)_{-\infty}^{\infty}$ (with $\mathbb{Z}$ as index set) which has the following monotonicity property: *there is a constant $c > 0$ such that $||(a_n)_n||_E \leq c||(b_n)_n||_E$ whenever $|a_n| < |b_n|$, $\forall n \in \mathbb{Z}$.*

Let $\vec{l}_p = (l_p, l_p(2^{-n}))$, $0 < p \leq \infty$ Then , a $\mathbb{Z}$-lattice E is called $\mathcal{K}$- non-trivial if

$$l_\infty(max(1, 2^{-n})) = \Delta(\vec{l}_\infty) \subseteq E.$$

This happens if

$$\sup_{||a||_E}(\sum_{n=-\infty}^{\infty}(min(1, 2^{-n})|a_n|)^p)^{1/p}.$$

Observe that the class of all $\mathbb{Z}$-lattice contains all interpolation spaces with respect to $\vec{l}_p$, $0 < p \leq \infty$.

We next define $\mathcal{K}$ and $\mathcal{J}$-spaces.

Definition 3.1

(i) Let E be a $\mathcal{K}$-non-trivial $\mathbb{Z}$-latice and $\overrightarrow{X}=(X_0,X_1)$ a locally convex couple. We define the $\mathcal{K}$-space $\overrightarrow{X}_{E,K}$ to consists of all $x\in\Sigma(\overrightarrow{X})$ such that $\mathcal{K}_{\alpha^0,\alpha^1}(2^n,x,\overrightarrow{X})\in E$ for all $(\alpha^0,\alpha^1)\in\in\mathcal{A}_{\prime}\times\mathcal{A}_1$.

(ii) Let E be a (p,j)-non-trivial $\mathbb{Z}$-lattice and $\overrightarrow{X}=(X_0,X_1)$ a locally convex couple. The $\mathcal{J}$-space $\overrightarrow{X}_{E,J}$ consist of all $x\in\Sigma(\overrightarrow{X})$ that may be written as

$$a=\sum_{n=-\infty}^{\infty}a_n, a_n\in\Delta(\overrightarrow{X}), \textit{converge in } \Sigma(\overrightarrow{X}) \textit{ with } (J(2^n,a_n,\overrightarrow{X}))_n\in E.$$

We put

$$p^{E,K}_{\alpha^0,\alpha^1}(x)=||(\mathcal{K}_{\alpha^0,\alpha^1}(2^n,x,\overrightarrow{X}))_n||_E,\ \text{for all } (\alpha^0,\alpha^1)\in\mathcal{A}_{\prime}\times\mathcal{A}_1.$$

Then the family of semi-norms $\left\{p^{E,K}_{\alpha^0,\alpha^1}\right\}_{(\alpha^0,\alpha^1)\in\mathcal{A}_{\prime}\times\mathcal{A}_1}$ generate a Hausdorf locally convex topology of $\overrightarrow{X}_{E,K}$. This topology is finner than that of $\Sigma(\overrightarrow{X})$ and less fine that of $\Delta(\overrightarrow{X})$.

Similarity we put

$$p^{E,J}_{\alpha^0,\alpha^1}(x)=\inf_{a=\Sigma_n a_n}||\mathcal{J}_{\alpha^0,\alpha^1}(2^n,a_n,\overrightarrow{X}))_n||_E,\ \text{for all } (\alpha^0,\alpha^1)\in\mathcal{A}_{\prime}\times\mathcal{A}_1.$$

Then the family of semi-norms $\left\{p^{E,J}_{\alpha^0,\alpha^1}\right\}_{(\alpha^0,\alpha^1)\in\mathcal{A}_{\prime}\times\mathcal{A}_1}$ generate a Hausdorf locally convex topology of $\overrightarrow{X}_{E,J}$. This topology is finner than that of $\Sigma(\overrightarrow{X})$ and less fine that of $\Delta(\overrightarrow{X})$.

When $E=l_p(2^{-n\theta}), 0<p\leq\infty, 0<\theta<1$ we recover the spaces $\overrightarrow{X}_{\theta,p,K}$ and $\overrightarrow{X}_{\theta,p,J}$.

Like in the case of spaces $\overrightarrow{X}_{\theta,p,K}$ and $\overrightarrow{X}_{\theta,p,J}$ we have

Theorem 3.1

Let $\overrightarrow{X}=(X_0,X_1)$ be a locally convex couple. Then

(i) The spaces $\overrightarrow{X}_{E,K}$ and $\overrightarrow{X}_{E,J}$ are interpolator with respect to $\overrightarrow{X}$;

(ii) If E is $\mathcal{K}$-and (p,J) non-trivial $\mathbb{Z}$-lattice

$$\overrightarrow{X}_{E,J}\hookrightarrow\overrightarrow{X}_{E,K};$$

(iii) If E *is* $\mathcal{K}$*-and* $(1,J)$ *non-trivial* $\mathbb{Z}$*-lattice and* $\overrightarrow{X}$ *is an Fréchet couple*

$$\overrightarrow{X}_{E,J}=\overrightarrow{X}_{E,K}\ .$$

Theorem 3.2

Let E_0*,* E_1 *and* E_2 *arbitrary* $\mathcal{K}$*-and* $(1,J)$*-non-trivial* $\mathbb{Z}$*-lattice. The following two conditions are equivalent:*

(i) for arbitrary Fréchet couples $\overrightarrow{X}=(X_0,X_1)$*,* $\overrightarrow{Y}=(Y_0,Y_1)$ *and* $\overrightarrow{Z}=(Z_0,Z_1)$ *and any bounded bilinear operator* $T: X_i\times Y_i\longrightarrow\longrightarrow Z_i,\ i=0,1,$ *we have*

$$T: X_{E_0,K}\times Y_{E_1,K}\longrightarrow Z_{E_2,K}.$$

(ii) the convolution $E_0 * E_1 \hookrightarrow E_2$

Acknowledgement

This paper was presented by Mr. Dan Lupulescu during The Fourteenth International Conference on Mathematics and its Applications (ICMA) which took place in Timişoara on 5-7 November 2015.

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