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NEW CONTRIBUTIONS IN A PROBLEM OF GEOMETRIC QUANTIZATION

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Abstract

Based on the study of the Manev's system, for which it is known that its geometric pre-quantization and Marsden- Weinstein switch, we propose to obtain the symplectic reduction switch with the horizontal polarization via the 1/2 correction forms. The proof follows a similar way as the Kostant's geometric quantization. [1]

Keywords and phrases: *1/2 correction forms; symplectic reduction; geometric quantization; horizontal polarization; Hilbert space.*

1 Introduction

In his book **[17]** Puta describes the connection between symplectic reduction and geometric prequantization at the cotangent level and its application in quantization of constrained mechanical systems. Forth in **[19]** is presented the symplectic reduction of the Manev Hamiltonian system and geometric prequantization, which proves that geometric prequantization and symplectic reduction are interchangeable processes.

More observations, remarks, problems and properties of geometric prequantization we find in ([**5**], [**27**]) and geometric quantization in ([**4**], **[8], [16], [18], [20], [21], [23], [26]**).

The basics of mechanics and quantum theory we have used in the works **[3]** and **[10]**.

Using our techniques in the papers of the above mentioned, we will show that in the 1\2 correction forms, the symplectic reduction switches with the geometric quantization in the case of horizontal polarization.

2 The Marsden-Weinstein reduction

Below are presented the levels of reduction depending on their generality: Poisson reduction, symplectic reduction and the cotangent manifold reduction (**[14]**).

[1]MSC (2010): 53D20; 53D50; 46E20

For *Poisson reduction* we start from a Poisson manifold $(P, \{\cdot,\cdot\}_P)$ and suppose that G is a Lie group acting Poisson on P, i.e.

$$\phi_g : P \to P,$$

is a Poisson diffeomorphism for any $g \in G$. Suppose further that the action is free and own. Then P/G has differential manifold structure, and the canonical projection

$$\pi : P \to P/G,$$

is a submersion.

Proposition 2.1. ([**12**], [**13**]) In the above assumptions, there is a unique structure of Poisson $\{\cdot,\cdot\}_{P/G}$ on P/G so π is an Poisson map.

If $H \in C^{\infty}(P, \mathbb{R})$ is a G - invariant Hamiltonian, it defines a canonical function $h \in C^{\infty}(P/G, \mathbb{R})$ through the relationship:

$$H = h \circ \pi.$$

Because π is a Poisson map, we have:

$$T\pi \cdot X_H = X_h \circ \pi,$$

or equivalent, the fields X_H and X_h are π-related (see [**2**]). We then say that X_H on P is reduced to a Hamiltonian field X_h on P/G or in other words, the integral curves of the X_H are projections (via π) on the integral curves of the X_h.

In the particular case $P = T^*G,\ \{\cdot,\cdot\}_P = \{\cdot,\cdot\}_{\omega}$, we have:

Proposition 2.2. (*Lie-Poisson reduction*) Suppose G acts on G by left translation and consider this lifted action on T^*G. Then

$$\left(P/G, \{\cdot,\cdot\}_{P/G}\right) \simeq \left(\mathcal{G}^*, \{\cdot,\cdot\}_-\right).$$

For *symplectic reduction* we start with symplectic manifold (M, ω). Let G a Lie group acting symplectic on M and having a momentum map J which is Ad^*-equivariant. Let the isotropy group

$$G_\mu = \{g \in G \mid g \cdot \mu = \mu\},$$

of $\mu \in \mathcal{G}^*$.

G_μ is invariant on $J^{-1}(\mu)$ and assuming that μ is a regular value of J shows that $J^{-1}(\mu)$ has differential manifold structure. If G_μ acts freely and properly on $J^{-1}(\mu)$, then the quotient space $M_\mu := J^{-1}(\mu)/G_\mu$ has a canonical structure of differential manifold. Then we have:

Proposition 2.3. (**[14]**, **[15]**) There is a unique symplectic structure ω_μ on M_μ so:

$$i_\mu^* \omega = \pi_\mu^* \omega_\mu,$$

where

$$i_\mu : J^{-1}(\mu) \to M,$$

is the application for inclusion and

$$\pi_\mu : J^{-1}(\mu) \to M_\mu,$$

is the canonical projection.

If $H \in C^\infty(M, \mathbb{R})$ is G-invariant then we define $H_\mu \in C^\infty(M_\mu, \mathbb{R})$ by

$$H = H_\mu \cdot \pi_\mu.$$

It can then easily be checked that the trajectories of X_H are beeing projected on the trajectories of X_{H_μ} or equivalent,.the fields X_H and X_{H_μ} are π-related.

Finally will discuss the *cotangent reduction*. The simplest case is reduced to 0, hence $\mu = 0$. Then we have:

$$\left((T^*Q)_{\mu=0}, \omega_0 \right) \simeq \left(T^*(Q/G), \omega_{T^*(Q/G)} \right).$$

Another important case is when G is abelian. In this case we have:

$$(T^*Q)_\mu \simeq T^*(Q/G),$$

but on $T^*(Q/G)$ we have canonical symplectic form plus a magnetic term (the curvature of the mechanical connection).

Abelian version of the reduction manifold cotangent is found in the works of Smale **[24]** and Satzer **[22]**. Neabelian version first appears in Abraham and Marsden **[1]**. Interpretation of the symplectic form on the (T^*Q) in terms of mechanical connection was given by Kummer **[9]**.

3 Real and complex polarizations

Definition 3.1. **[17]** A distribution D on M is a map that assigns to each point $x \in M$ a linear subspace $D_x \subset T_x M$ so the following conditions to be checked:

(i) $k = \dim D_x, \ \forall x \in M$;

(ii) $\forall x_0 \in M, \ \exists \ V_{x_0} \in \mathcal{V}_{(x_0)}, \ \exists \ X_1, ..., X_n \in \mathfrak{X}(\mathcal{V}_{x_0})$ linearly independent so

$$Span \ \left((X_1)_x, ..., (X_n)_x \right) = D_x, \ \forall x \in V_{x_0}.$$

Definition 3.2. **[17]** A distribution D is integrable if for any $x \in M$, there is a submanifold N of M such that:

(i) $x \in N$;

(ii) $\dim\ N = k$;

(iii) $T_y N = D_y,\ \forall y \in N$.

Proposition 3.1. **(***Frobenius***)** D is integrable if and only if for any $X, Y \in \mathfrak{X}(M, D)$ result that $[X, Y] \in \mathfrak{X}(M, D)$, where

$$\mathfrak{X}(M, D) = \{X \in \mathfrak{X}(M) \mid X_x \in D_x,\ \forall x \in M\}.$$

Definition 3.3. **[17]** An integrable distribution is called foliation. The maximal submanifolds $N \subset M$ for which $T_x N = D_x$ are called foliation leaves.

Let M/D the space of foliation leaves D. If there is a differentiable structure on M/D so the canonical projection $\pi : M \to M/D$, to be a surjective submersion, then D is called reducible foliation.

Example 3.1.[17] *(Vertical distribution)* Let $M = T^*Q$ with the local canonical coordinates

$$\left\{q^1, ..., q^n, p_1, ..., p_n\right\}.$$

D^v distribution generated by the vector fields $\dfrac{\partial}{\partial p_1}, ..., \dfrac{\partial}{\partial p_n}$, hence

$$D^v = Span\left(\frac{\partial}{\partial p_1}, ..., \frac{\partial}{\partial p_n}\right),$$

is called the vertical distribution on M. Obvious that it is integrable, being local data leaves by:

$$\begin{cases} q^1 = \text{constant}, \\ \quad\ \ ... \\ q^n = \text{constant}. \end{cases}$$

Because

$$D_{\diagup D^v} \simeq Q;$$

It follows that D^v is a reducible foliation.

Example 3.2. **[17]** *(Horizontal distribution)* Let $M := T^*Q$ with the local canonical coordinates

$$\left\{q^1, ..., q^n, p_1, ..., p_n\right\}.$$

D^h distribution generated by the vector fields $\dfrac{\partial}{\partial q^1}, ..., \dfrac{\partial}{\partial q^n}$, hence

$$D^v = Span\left(\frac{\partial}{\partial q^1}, ..., \frac{\partial}{\partial q^n}\right),$$

is called the horizontal distribution on M. Obvious that it is integrable, being local data leaves by:

$$\begin{cases} p_1 = \text{constant}, \\ \quad\ \ldots \\ p_n = \text{constant}. \end{cases}$$

Because

$$D_{\diagup D^h} \simeq Q,$$

it follows that it is reducible.

Definition 3.4. **[17]** Let (M, ω) be a symplectic $2n$-dimensional manifold. A real polarization on M is a foliation D on M such that D is the maximal isotropic with respect to ω i.e., the following conditions are satisfied:

(i) $\omega_x(D_x, D_x) = 0, \ \forall x \in M$;

(ii) no other subspace of T_xM which contains D_x does not have the property (i), or in other words D_x is maximal with the property (i).

In particular, $\dim D_x = n, \ \forall x \in M$. Polarization D is called reducible if the foliation D is reducible.

Example 3.3. Let $(M, \omega) := (T^*Q, \omega = d\theta)$. Then check easily that D^v and D^h are real polarizations, reductibles on M.

A characterization of the real polarizations is given by:

Proposition 3.2. **([17], [26])** Let (M, ω) be a symplectic $2n$-dimensional manifold. Then a distribution D on M is a real polarization on M if and only if for any $x \in M$, there is $U_x \in \mathcal{V}(x)$ and independent functions $f_1, \ f_2, \ \ldots, \ f_n \in C^\infty(U_x, \mathbb{R})$, such that the following conditions are met:

$$(i) \ \ D_y = Span \ \left(X_{f^1}(y), \ldots, X_{f^n}(y)\right), \ \forall y \in U_x;$$

$$(ii) \qquad \left\{f^i, f^j\right\}_\omega = 0 \text{ on } U_x, \ \forall i, j = 1, 2, \ldots n.$$

Unfortunately there are real situations when on a symplectic manifold there does not exist real polarization. For example this is the case of 2-dimensional sphere from $\mathbb{R}^3$, where there is no field of continuous and nonzero vectors on S^2 and so we do not have a real polarization on S^2. We are so led to the following definition:

Definition 3.5. **[17]** Let (M, ω) be a symplectic $2n$-dimensional manifold. A complex polarization on M is a complex distribution P on M such that:

(i) For $\forall x \in M, \ P_x$ is a Lagrangean subspace of T_xM^C, complexified of T_xM, and the Lagrangean means that is maximal isotropic;

(ii) $D_x := P_x \cap \overline{P_x} \cap T_xM$ has $\dim D_x = k, \ \forall x \in M$;

(iii) P and $P + \overline{P}$ are closed in relation to the Lie commutator.

P is called reducible if M/D is a differential manifold structure such that $\pi : M \to M/D$ is surjective submersion.

P is called positive if $-i\omega\left(X, \overline{X}\right) \geq 0, \ \forall X \in \mathfrak{X}(M, P)$.

Example 3.4. Let $(M,\omega) := \left(T^*\mathbb{R} \simeq \mathbb{R}^2, dp \wedge dq\right)$ and P complex distribution on M given by

$$P := \mathbb{C}\left(\frac{\partial}{\partial p} + i\frac{\partial}{\partial q}\right) = \mathbb{C}\ X_{p+iq}.$$

Then check easily that:
(i) $\omega(P,P) = 0$;
(ii) $\dim P = 1$;
(iii) $P = \overline{P} = \{0\} \Rightarrow M_{/D} \simeq M$;
(iv) P and $\overline{P}$ are closed in relation to the Lie commutator.
Therefore P is a reducible complex polarization on the M.

Example 3.5. Let $(M,\omega) := \left(T^*\mathbb{R}^2 \simeq \mathbb{R}^4, dp_1 \wedge dq^1 + dp_2 \wedge dq^2\right)$ and

$$P := \mathbb{C}\left(\frac{\partial}{\partial p_1} + i\frac{\partial}{\partial q^1}\right) \oplus \mathbb{C}\frac{\partial}{\partial p_2} = \mathbb{C}\ X_{p_1+iq^1} \oplus \mathbb{C}\frac{\partial}{\partial q^2}.$$

Then we have:
(i) $\omega(P,P) = 0$;
(ii) $\dim P = 2$;
(iii) $P = \overline{P} = \{0\} \Rightarrow M_{/D} \simeq M$;
(iv) P and $P + \overline{P}$ are closed in relation to the Lie commutator.
Therefore P is a complex polarization and reducible on M.

4 The symplectic reduction and the geometric quantization of Manev's Hamiltonian system

The planets do not follow Kepler's first law, i.e. their orbits do not describe the ellipses thanks to the corrections derived from general relativity **[25]**.

Gravitational potential takes the following form

$$V(r) = \frac{k}{r} + \frac{A}{r^2},$$

to a first approximation to the shape correction $\frac{A}{r^2}$ for some constant A.

For the first time the potential of this form was introduced by Newton and Clairaut. In 1924 a Manev **[11]** such potential as a correction to the classic Newtonian potential used in celestial mechanics.

Manev's Hamiltonian system is the triple

$$\left(T^*\mathbb{R}^3, dp_r \wedge dr + dp_\theta \wedge d\theta + dp_\varphi \wedge d\varphi, \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\varphi^2}{2mr^2\sin^2\theta} - V(r)\right),$$

where $\{r, \theta, \varphi\}$ are the spherical coordinates on $\mathbb{R}^3$, $\{p_r, p_\theta, p_\varphi\}$ are conjugate moments corresponding and

$$\gamma = \frac{3k}{c^2},\ A = \frac{k\gamma}{2},\ k = GM,\ m = \frac{M}{M+1},$$

where G is the gravitational constant, c is the speed of light, M is the mass of the particle at the origin.

In **[19]** is made the symplectic reduction of Manev's Hamiltonian.system where it is easy to see that the system is invariant under the lifted action of $SO\,(3)$ to $T^*\mathbb{R}^3$. The action is Hamiltonian with the equivariant momentum map $J : T^*\mathbb{R}^3 \to (so\,(3))^* \simeq \mathbb{R}^3$,

$$J\,(r, \theta, \varphi, p_r, p_\theta, p_\varphi) := (-p_\varphi \cos\varphi \cot\theta - p_\theta \sin\varphi, p_\theta \cos\varphi - p_\varphi \cot\theta \sin\varphi, p_\varphi)\,.$$

We have found (**[19]**) that the reduced Hamiltonian is

$$H_\mu\,(r, p_r) = \frac{p_r^2}{2m} + \frac{A}{r^2} + \frac{l^2 - 2mA}{2mr^2} - V\,(r)\,,$$

where we have considered without losing generality as $\mu \in \mathbb{R}^3,\ \mu \neq 0$ be a regular value of J and

$$J^{-1}\,(\mu) = \left\{ \left(r, \frac{\pi}{2}, \varphi, p_r, 0, l \right) \in T^*\mathbb{R}^3 \mid \mu = (0,0,l)\,,\ l \neq 0 \right\}.$$

Further, it was observed that $M_\mu = J^{-1}\,(\mu) \diagup G_\mu$ (**[19]**), where the coadjoint isotropy subgroup G_μ of μ is isomorphic to $SO\,(2)$ and acts on $J^{-1}\,(\mu)$ as

$$G_\mu \times J^{-1}\,(\mu) \to J^{-1}\,(\mu)\,,$$

and

$$\left(e^{i\alpha}, \left(r, \frac{\pi}{2}, \varphi, p_r, 0, l \right) \right) \mapsto \left(r, \frac{\pi}{2}, \varphi + \alpha, p_r, 0, l \right).$$

Finally because in spherical coordinates ω is given by $\omega = dp_r \wedge dr + dp_\theta \wedge d\theta + dp_\varphi \wedge d\varphi$ result as $\omega_\mu = dp_r \wedge dr$, hence the reduced Manev's Hamiltonian system (**[19]**) is the triple

$$(M_\mu, \omega_\mu, H_\mu)\,.$$

In the above result:

Proposition 4.1. **[19]** $i_\mu^*\theta = \pi_\mu^*\theta_\mu$ where $\theta = p_r dr + p_\theta d\theta + p_\varphi d\varphi$ and $\theta_\mu = p_r dr$.

Let $(\mathcal{H}^\omega, \delta^\omega)$ be the Hilbert representation space and the prequantum operator on $\left(T^*\mathbb{R}^3, \omega = d\theta\right)$, resp.$(\mathcal{H}^{\omega_\mu}, \delta^{\omega_\mu})$, $\left(T^*\mathbb{R}_+^*, \omega_\mu = d\theta_\mu\right)$, given by geometric prequantization.

Proposition 4.2. If (M, ω) is an exact symplectic manifold and $\omega = d\theta$, then it is quantizable and we get:

$$\nabla_X^{\omega} f = X(f) - \frac{i}{\overline{h}}\theta(X) f,$$

$$L^{\omega} = (M \times \mathbb{C}, pr_1, M),$$
$$\Gamma(L^{\omega}) = C^{\infty}(M, \mathbb{C}),$$
$$((x, z_1), (x, z_2))_x = z_1 \overline{z}_2,$$
$$\mathcal{H}^{\omega} = L^2(M, \mathbb{C}),$$
$$\delta_f^{\omega} = f - i\overline{h}\left[X_f - \frac{i}{\overline{h}}\theta(X_f)\right].$$

Proposition 4.3. [19] Let $C^{\infty}_{SO(3)}\left(T^*\mathbb{R}^3, \mathbb{R}\right)$ be the space of $SO(3)$-invariant, real valued functions defined on $T^*\mathbb{R}^3$. Then for each $f \in C^{\infty}_{SO(3)}\left(T^*\mathbb{R}^3, \mathbb{R}\right)$ and for each $g \in L^2_{SO(3)}\left(\mathbb{R}^*_+, \mathbb{C}\right)$ we obtain:

(i) The Hilbert spaces $L^2_{SO(3)}\left(T^*\mathbb{R}^*_+, \mathbb{C}\right)$ and $\left(L^2_{SO(3)}\left(T^*\mathbb{R}^3, \mathbb{C}\right)\right)_{\mu}$ are isomorphic;

(ii) $\delta^{\omega_\mu}_{f_\mu}(g_\mu) = \left[\delta^{\omega}_f(g)\right]_{\mu}$, or in other words geometric prequantization and symplectic reduction are interchangeable processes.

5 The 1/2 correction forms in the symplectic reduction

Considering (M, ω) a symplectic manifold of dimension $2n$ and a complex polarization P on M. Then $\wedge^n P^0$ is a complex line bundle on M noted K_P where P^0 is the annihilating of P, i.e. consists of 1-forms on M to cancels on P, i.e. $\omega(X) = 0, \; \forall \; X \in \mathcal{X}(M, P)$. P^0 and P are canonically identify, via application $v \in P \mapsto i_v\omega \in P^0$ ([7]).

Definition 5.1. [17] Complex line bundle K_P constructed above is called canonical bundle of M.

Definition 5.2. [17] M is called metaplectic manifold if there is a subbundle (denoted by) $N_P^{\frac{1}{2}}$ of K_P so that $N_P^{\frac{1}{2}} \times N_P^{\frac{1}{2}} = K_P$.

Proposition 5.1. [17] (M, ω) is metaplectic in relation to a complex polarization P if and only if the first Chern class of K_P is 0 modulo 2.

In the previous sentence hypotheses, $H^1(M,\mathbb{Z}_2)$ is parametrizing the set of bundles $N_P^{\frac{1}{2}}$.

Let (M,ω) be a metaplectic and quantizable manifold in relation to a complex polarization reducible P. Then we define the following spaces:

$$(i)\ \Gamma_P\left(L^\omega\otimes N_P^{\frac{1}{2}}\right)\{s\otimes\mu\in\Gamma\left(L^\omega\otimes N_P^{\frac{1}{2}}\right)\mid\nabla_X^\omega s=0 \text{ and } L_X\mu=0,\ \forall X\in\mathcal{X}(M,P)\},$$

$$(ii)\ P\mathcal{H}^{\frac{1}{2}}=\left\{s\otimes\mu\in\Gamma_P\left(L^\omega\otimes N_P^{\frac{1}{2}}\right)\mid\int\limits_{\frac{M}{D}}(s\otimes\mu,s\otimes\mu)<\infty\right\},$$

$$(iii)\ \mathcal{H}^{\frac{1}{2}} \text{ is completion of } P\mathcal{H}^{\frac{1}{2}},$$

$$(iv)\ C^\infty(M,P;\mathbb{R}) \text{ - space of quantizable functions on } M.$$

Let the operator

$$(\delta_P^\omega)_f^{\frac{1}{2}}(s\otimes\mu)\overset{\text{def}}{=}-i\hbar\left(\nabla_{X_f}^\omega s\right)\otimes\mu-i\hbar s\otimes L_X\mu+fs\otimes\mu,\ \ \forall f\in C^\infty(M,P;\mathbb{R}),\forall s\otimes\mu\in P\mathcal{H}^{\frac{1}{2}}.$$

Proposition 5.2. [8] The pair $\left(\mathcal{H}^{\frac{1}{2}},(\delta_P^\omega)^{\frac{1}{2}}\right)$ defines a pre-quantization of $C^\infty(M,P;\mathbb{R})$, *i.e.* on this algebra (the algebra structure induced Poisson structure given by ω) are met the Dirac conditions, i.e. for $\forall f,g\in C^\infty(M,P;\mathbb{R})$ and $\forall\alpha\in\mathbb{R}$, we have

$$(i)\ (\delta_P^\omega)_{f+g}^{\frac{1}{2}}=(\delta_P^\omega)_f^{\frac{1}{2}}+(\delta_P^\omega)_g^{\frac{1}{2}},$$

$$(ii)\ (\delta_P^\omega)_{\alpha f}^{\frac{1}{2}}=\alpha(\delta_P^\omega)_f^{\frac{1}{2}},$$

$$(iii)\ (\delta_P^\omega)_{1_M}^{\frac{1}{2}}=Id_{\mathcal{H}^{\frac{1}{2}}},$$

$$(iv)\ \left[(\delta_P^\omega)_f^{\frac{1}{2}},(\delta_P^\omega)_g^{\frac{1}{2}}\right]=\frac{1}{i\hbar}(\delta_P^\omega)_{\{f,g\}_\omega}^{\frac{1}{2}}.$$

Proposition 5.3. **[17]** Let $\mathcal{H}^{\frac{1}{2}}$ be a completion of $P\mathcal{H}^{\frac{1}{2}}$. Then $\mathcal{H}_{D^v}^{\frac{1}{2}}\simeq L^2(\mathbb{R},\mathbb{C})$ (are isomorphic).

Remark 5.1. For D_h it can be made an analogous construction and furthermore, using the Fourier transform, it can be shown that $\mathcal{H}_{D^v}^{\frac{1}{2}}$ and $\mathcal{H}_{D^h}^{\frac{1}{2}}$ are isomorphic i.e.

$$\mathcal{H}_{D^v}^{\frac{1}{2}}\simeq\mathcal{H}_{D^h}^{\frac{1}{2}}$$

Proposition 5.4. **[6]** For the 1-dimensionl oscillator harmonic we find the result given in classical Schrödinger quantization.

Let G an abelian Lie-gruop which acts freely and properly on Q (a smooth orientable n-dimensional manifold), $\mu \in \mathcal{G}^*$ a regular value of the momentum mapping $J : T^*Q \to \mathcal{G}^*$ such that

$$i_\mu^* \theta = \pi_\mu^* \theta_\mu$$

and D^h be the horizontal polarization on T^*Q.

We denote by $\left(\mathcal{H}_{D^h}^{1/2}, \delta_{D^h}^{1/2}\right)$ the Hilbert representation space and the quantum operator resp. $\left(\mathcal{H}_{D_\mu^h}^{1/2}, \delta_{D_\mu^h}^{1/2}\right)$ the same.

Following the (**[17], [19]**) then we will show below that:

Theorem 5.1. Let $\left(\mathcal{H}_{D^h}^{1/2}\right)_G$ be the Hilbert space of G invariant states, i.e.

$$\left(\mathcal{H}_{D^h}^{1/2}\right)_G = \left\{ f \in \mathcal{H}_{D^h}^{1/2} \mid f \text{ is } G \text{ - invariant} \right\}$$

and $C_G^\infty\left(T^*Q, D^h; \mathbb{R}\right)$ the space of G - invariant quantizable functions on T^*Q. Then for each $f \in C_G^\infty\left(T^*Q, D^h; \mathbb{R}\right)$ and $g \in \left(\mathcal{H}_{D^h}^{1/2}\right)_G$ we have:

$$\left(\delta_{D_\mu^h}\right)_{f_\mu}(g_\mu) = \left[(\delta_{D^h})_f(g)\right]_\mu,$$

or, in other words, the geometric quantization in horizontal polarization and symplectic reduction are interchangeable processes, in 1/2 correction forms.

Proof.

We first check the condition and we start from the equality

$$(f \cdot g)(i_\mu(x)) = f(i_\mu(x)) \cdot g(i_\mu(x)),$$

but

$$f_\mu(\pi_\mu(x)) = f(i_\mu(x)),$$

then we have

$$\begin{aligned} f(i_\mu(x)) \cdot g(i_\mu(x)) &= f_\mu(\pi_\mu(x)) \cdot g_\mu(\pi_\mu(x)) \\ f_\mu(\pi_\mu(x)) \cdot g_\mu(\pi_\mu(x)) &= [f_\mu \cdot g_\mu](\pi_\mu(x)) \\ (f \cdot g)(i_\mu(x)) &= [f_\mu \cdot g_\mu](\pi_\mu(x)). \end{aligned}$$

The following condition are true

$$
\begin{aligned}
X_f(g)(i_\mu(x)) &= (X_f)_x(g \circ i_\mu) \\
(X_f)_x(g \circ i_\mu) &= (X_f)_x(g_\mu \circ \pi_\mu) \\
(T_x \pi_\mu)(X_f)_x(g_\mu) &= (X_f)_x(g_\mu \circ \pi_\mu) \\
\left(X_{f_\mu}\right)_{\pi_\mu(x)}(g_\mu) &= (T_x \pi_\mu)(X_f)_x(g_\mu) \\
X_{f_\mu}(g_\mu)(\pi_\mu(x)) &= \left(X_{f_\mu}\right)_{\pi_\mu(x)}(g_\mu) \\
X_f(g)(i_\mu(x)) &= X_{f_\mu}(g_\mu)(\pi_\mu(x)).
\end{aligned}
$$

The last condition yields

$$
\begin{aligned}
[\theta(X_f) \cdot g](i_\mu(x)) &= [\theta(X_f)(i_\mu(x))] \cdot [g(i_\mu(x))] \\
[\theta(X_f)(i_\mu(x))] \cdot [g(i_\mu(x))] &= \left[\theta_{i_\mu(x)}(X_f)_{i_\mu(x)}\right] \cdot [g(i_\mu(x))] \\
\left[\theta_{i_\mu(x)}(X_f)_{i_\mu(x)}\right] \cdot [g(i_\mu(x))] &= \left[\left(\pi_\mu^* \theta_\mu\right)_{\pi_\mu(x)}(X_f)_x\right] \cdot [g(i_\mu(x))] \\
\left[\left(\pi_\mu^* \theta_\mu\right)_{\pi_\mu(x)}(X_f)_x\right] \cdot [g(i_\mu(x))] &= \left[(\theta_\mu)_{\pi_\mu(x)}\left(T_x \pi_\mu (X_f)_x\right)\right] \cdot [g(i_\mu(x))] \\
\left[(\theta_\mu)_{\pi_\mu(x)}\left(T_x \pi_\mu (X_f)_x\right)\right] \cdot [g(i_\mu(x))] &= \left[(\theta_\mu)_{\pi_\mu(x)}\left(X_{f_\mu}\right)_{\pi_\mu(x)}\right] \cdot [(g \circ i_\mu)(x)] \\
\left[(\theta_\mu)_{\pi_\mu(x)}\left(X_{f_\mu}\right)_{\pi_\mu(x)}\right] \cdot [(g \circ i_\mu)(x)] &= \left[\theta\left(X_{f_\mu}\right)(\pi_\mu(x))\right] \cdot [g_\mu(\pi_\mu(x))].
\end{aligned}
$$

From the above we have proved that :

$$
\begin{aligned}
(f \cdot g)(i_\mu(x)) &= [f_\mu \cdot g_\mu](\pi_\mu(x)) \\
X_f(g)(i_\mu(x)) &= X_{f_\mu}(g_\mu)(\pi_\mu(x)) \\
[\theta(X_f) \cdot g](i_\mu(x)) &= \left[\theta\left(X_{f_\mu}\right)(\pi_\mu(x))\right] \cdot [g_\mu(\pi_\mu(x))]
\end{aligned}
$$

hence in other words we have shown that and in 1/2 correction forms, horizontal polarization and symplectic reduction are interchangeable processes.

From the above we have:

Proposition 5.5. The Hilbert spaces $\left(\mathcal{H}^{1/2}_{D^h_\mu}\right)_G$ and $\left[\left(\mathcal{H}^{1/2}_{D^h}\right)_G\right]_\mu$ are isomorphic.

Proposition 5.6. $f_\mu \in C^\infty_G\left(T^*Q, D^h_\mu; \mathbb{R}\right)$ and $g_\mu \in \left(\mathcal{H}^{1/2}_{D^h_\mu}\right)_G$.

Remark 5.2. You can check immediately as in the case of vertical polarizations, 1/2 correction forms give the same result.

Remark 5.3. At the cotangent level in **[17]** shall describe the connection between symplectic reduction and geometric quantization.

6 Conclusions

The 1\2 correction forms has a very important role, as we have seen in the case of one dimensional harmonic oscillator **[7]** and the case of symmetric free rigid body [**6**]. The novelty of this article lies in the fact that the 1\2 correction forms, the symplectic reduction switch with horizontal polarization.

References

[1] R. Abraham and J. Marsden., Foundations of Mechanics, 2nd ed., *(Addison-Wesley, 1985).*

[2] M. Craioveanu, Introducere in geometrica diferentiala, (Editura Universitatii de Vest, Timisoara, 2008). (in Romanian).

[3] P. A. M. Dirac, The Principles of Quantum Mechanics, 4th ed., (Oxford University Press, 1976).

[4] M. J. Gotay, Contrains, reduction and quantization, J. Math. Phys., 27 (8), (1986), 2051-2066.

[5] C. Hedrea, I. Golet, R. Negrea, Geometric prequantization of the dual a Lie algebra, International Journal of Geometric Methods in Modern Physics, 10(6), (2013), 1320009.

[6] C. Hedrea, R. Negrea, I. Zaharie, The 1/2 correction forms in geometric quantization of the symmetric free rigid body, International Journal of Geometric Methods in Modern Physics, 9(7), (2012), 1220011.

[7] C. Hedrea, R. Negrea, I. Zaharie and M. Puta$^+$, On a problem of geometric quantization, International Journal of Geometric Methods in Modern Physics,8(6), (2011), 1259-1268.

[8] B. Kostant, Quantization and unitary representations, Lecture Notes in Mathemetics, 170, (1970), 87-208.

[9] M. Kummer, On the construction of the reduced phase space of a Hamiltonian system with symmetry, Indiana Univ. Math Journal, 30, (1981), 281-291.

[10] K. S. Lam, Non-Relativistic Quantum Theory - Dynamics, Symmetry and Geometry (World Scientific, California), (2009).

[11] G. Manev, La gravitation et le principle de l'egalite de l'action et de la reaction, Comp. Rend., 178, (1924), 2159-2161.

[12] J. Marsden, Lectures on Mechanics, London Mathematical Society Lecture Notes Series, 174, (1992), Cambridge University Press.

[13] J. Marsden and T. Ratiu, Introduction to Mechanics and Symmetry, Texts in Applied Mathematics, 17, (1994), Springer-Verlag, Berlin, Heidelberg, New York.

[14] J. Marsden and A. Weinstein, Reduction of symplectic manifolds with symmetry, Rep. Math. Phys., 5, (1974), 121-130.

[15] K. Meyer, Symmetries and integrals in mechanics, in Dynamical Systems, ed. M. Peixoto, (Academic Press, 1973), 259-173.

[16] I.M. Mladenov, Reductions and Quantization, International Journal of Theoretical Physics, 28(10), (1989), 1255-1267.

[17] M. Puta, Hamiltonian Mechanical Systems and Geometric Quantization, Mathematics and its Applications, 260, (Springer-Verlag, Berlin, 1993).

[18] M. Puta, C. Hedrea, Some remarks on geometric quantization, in Proc. 9th National Conf. of the Romanian Mathematical Society (Lugoj, Romania, 2005), 348-353.

[19] M. Puta, C. Hedrea, Some remarks on Manev's Hamiltonian system, Tensor, N.S., 66(1), (2005), 71-75.

[20] M. Puta, Gh. Ivan, Groupoids, Lie-Poisson structures and quantization, Proceedings of the International Conference on Group Theory, Timisoara, September 17-20, Analele Univ. Timisoara, 39, (1992), Special Issue, 117-123.

[21] W.G. Ritter, Geometric Quantization, preprint, Jefferson Physical Laboratory, Harvard University Cambridge, MA (2008), 1-10.

[22] W. Satze, Canonical reduction of mechanical systems invariant under abelian group actions with an application to celestial mechanics, Indiana Univ. Math. Journal, 26, (1977), 951-976.

[23] I. Segal, Quantization of nonlinear systems, Journ. Math. Phys., 1, (1960), 468-488.

[24] S. Smale, Topology and mechanics I, and II, Inventiones Mathematicae, 10, 305-331, **11**, (1970), 45-64.

[25] J.M. Souriau, Structure des systemes dynamiques, (Dunod,(1970), Paris).

[26] N.M.J. Woodhouse, Geometric Quantization (Oxford University Press, 1992).

[27] A. Weinstein and M. Zambon, Variations on prequantization, Travaux Math., 16, (2005), 187-219.

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